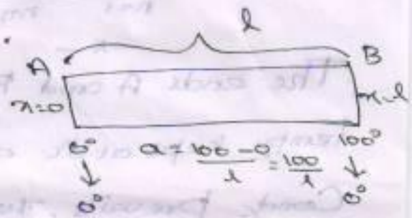
A rod of length l has its ends A and B kept at $0^\circ c$ and $100^\circ c$ resp. Until steady state conditions prevail. If the temp. at B is reduced suddenly to $0^\circ c$ and kept so while that of A is maintained find the temperature $u(x,t)$.



Sol:

The temp. $u(x,t)$ is from

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$$

The condn. are

(1) $u(0,t)=0$ (2) $u(l,t)=0$ (3) $u(x,0)=ax+b=\frac{100}{l}x$.

The suitable sol. is

$$u(x,t)=(A\cos px+B\sin px)\,Ce^{-\alpha^2p^2t}$$

Apply (1) $x=0$

$$A(Ce^{-\alpha^2p^2t})=0 \Rightarrow A=0$$

Apply (2)

$$u(l,t)=\cancel{A\cos pl}(B\sin pl)(Ce^{-\alpha^2p^2t})=0$$

$$\sin pl=0=\sin n\pi \Rightarrow pl=n\pi \quad \boxed{p=n\pi/l}$$

$$u(x,t)=B\sin\frac{n\pi x}{l}\,Ce^{-\frac{\alpha^2n^2\pi^2t}{l^2}}$$

The most gen. $u(x,t)=\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)e^{-\frac{\alpha^2n^2\pi^2t}{l^2}}$ —(4)

Apply (3) in (4)

$$u(x,0)=\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

$$C_n=\frac{2}{l}\int_0^l \frac{100x}{l}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$=\frac{200}{l^2}\int_0^l x\sin\left(\frac{n\pi x}{l}\right)dx$$

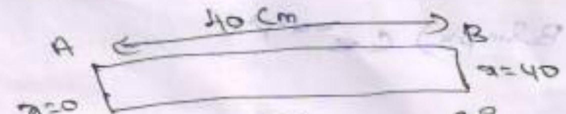
$$=\frac{200}{l^2}\left[x\left(-\cos\left(\frac{n\pi x}{l}\right)\left(\frac{l}{n\pi}\right)\right)+\frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{200}{l^2}\left[-\frac{l^2}{n\pi}(-1)^n\right] = \frac{-200(-1)^n}{n\pi}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{-200(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{l}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{l^2}}$$

—x—

The ends A and B of a rod 40cm long have their temp. kept at 0°C and 80°C resp. until steady state cond. prevails. The temp. of the end B is then suddenly reduced to 40°C and kept so, while that of the end A is kept at 0°C find the subsequent temp. distribution u(x,t) in the rod.



$0°$ $\quad a = \frac{80-0}{40} = 2$ $\quad 80°$

$b = 0$

$0°$ $\quad ax+b = 2x$ $\quad 40°$

$$\frac{40-0}{40} = 1$$

$ax+b = x.$

The temp. $u(x,t)$ is from $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

The cond. are

(1) $u(0,t)=0$ (2) $u(40,t)=40$ (3) $u(x,0)=ax+b=2x.$

The suitable soln.

$$u(x,t) = (A\cos px + B\sin px)\, C e^{-\alpha^2 p^2 t} + U_s(x)$$

$$U_s(x) = ax+b = x.$$

$$u(x,t) = x + (A\cos px + B\sin px)\, C e^{-\alpha^2 p^2 t} \quad \text{—(1)}$$

Apply (1)

$A\,(C e^{-\alpha^2 p^2 t}) \Rightarrow A = 0$

Apply ②

$$40 = 40 + (B \sin 40p) Ce^{-\alpha^2 p^2 t}$$

$$(B \sin 40p) Ce^{-\alpha^2 p^2 t} = 0$$

$$\sin 40p = \sin n\pi \quad p = n\pi/40.$$

$$u(x,t) = x + B \sin\left(\frac{n\pi x}{40}\right) Ce^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$

Most gen. sol. is

$$u(x,t) = x + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$

Apply (3) put $t=0$

$$2x = x + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right)$$

$$x = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right)$$

$$C_n = \frac{2}{40} \int_0^{40} x \sin\left(\frac{n\pi x}{40}\right) dx$$

$$= \frac{1}{20} \left[x \left(-\cos\left(\frac{n\pi x}{40}\right) \left(\frac{40}{n\pi}\right) \right) + \sin\left(\frac{n\pi x}{40}\right) \left(\frac{1600}{n^2\pi^2}\right) \right]_0^{40}$$

$$= \frac{1}{20} \left[-\frac{x\, 40}{n\pi} \cos\left(\frac{n\pi x}{40}\right) + \frac{1600}{n^2\pi^2} \sin\left(\frac{n\pi x}{40}\right) \right]_0^{40}$$

$$= \frac{1}{20} \left[-\frac{1600}{n\pi} (-1)^n \right] = -\frac{80 (-1)^n}{n\pi}$$

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{-80 (-1)^n}{n\pi} \sin\left(\frac{n\pi x}{40}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$