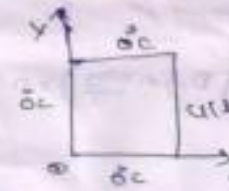


TOPIC 10: STEADY STATE SOLUTIONS OF TWO DIMENSIONAL HEAT EQUATION

Finite Plate with Value Given in y-direction.

- ① $u(m, 0) = 0$
- ② $u(m, l) = 0$
- ③ $u(0, y) = 0$
- ④ $u(l, y) = f(y)$

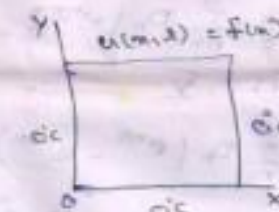


The Suitable Sol.

$$u(m, y) = (Ae^{py} + Be^{-py}) (C \cos py + D \sin py)$$

x-Direction

- ① $u(0, y) = 0$
- ② $u(l, y) = 0$
- ③ $u(m, 0) = 0$
- ④ $u(m, l) = f(m)$

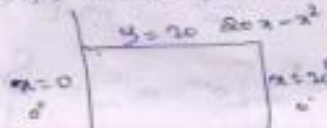


The Suitable Sol.

$$u(m, y) = (A \cos py + B \sin py) (C e^{px} + D e^{-px})$$

A Sq Plate is bounded by the lines $x=0, y=0, x=20$ and $y=20$. Its faces are insulated. The temp. along the upper horizontal edge is given by $u(m, 20) = 2(20-m) + 20x - x^2$ $0 \leq x \leq 20$ while the other edges are kept at 0°C . Find the Steady state temp. $u(x, y)$ in the plate.

Sol:





The temp. $u(x,y)$ is from

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The Cond. are

① $u(x,0) = 0$ ② $u(20,y) = 0$ ③ $u(x,20) = 0$ ④ $u(0,y) = 20x - x^2$

The suitable soln. is

$$u(x,y) = [A \cos px + B \sin px] [C e^{py} + D e^{-py}]$$

Apply ① $0 = A(C e^{0y} + D e^{-0y}) \Rightarrow A = 0$

Apply ② $0 = B \sin(20p) (C e^{py} + D e^{-py})$
 $B \sin(20p) = 0 \Rightarrow 20p = n\pi \Rightarrow p = \frac{n\pi}{20}$

Apply ③

$$0 = B \sin px (C + D)$$

$$C + D = 0 \Rightarrow D = -C$$

\therefore ① becomes

$$u(x,y) = B \sin px C [e^{py} - e^{-py}]$$

$$= B \sin\left(\frac{n\pi x}{20}\right) C [e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}]$$

Now gen. soln.

$$u(x,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{20}\right) [e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}}]$$

Apply ④

$$20x - x^2 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{20}\right) [e^{n\pi} - e^{-n\pi}]$$

$$C_n [e^{n\pi} - e^{-n\pi}] = \frac{2}{20} \int_0^{20} (20x - x^2) \sin\left(\frac{n\pi x}{20}\right) dx$$

$$= \frac{1}{10} \left[\frac{2x(20-x^2)}{n\pi} \cos\left(\frac{n\pi x}{20}\right) + \frac{400}{n^3\pi^3} (20-2x) \sin\left(\frac{n\pi x}{20}\right) - \frac{1600}{n^5\pi^5} \cos\left(\frac{n\pi x}{20}\right) \right]_0^{20}$$

$$= \frac{1}{10} \left[\frac{-1600}{n^3\pi^3} (-1)^n + \frac{16000}{n^3\pi^3} (1) \right]$$

$$= \frac{1600}{n^3\pi^3} [-(-1)^n + 1]$$

$$= \begin{cases} 0 & n = 2, 4, 6, \dots \\ \frac{3200}{n^3\pi^3} & n = 1, 3, 5, \dots \end{cases}$$

$$C_n = \frac{3200}{n^3\pi^3} (e^{n\pi} - e^{-n\pi})$$

$$u(x,y) = \sum_{n=1,3,5,\dots}^{\infty} \frac{3200}{n^3\pi^3} \sin\left(\frac{n\pi x}{20}\right) (e^{\frac{n\pi y}{20}} - e^{-\frac{n\pi y}{20}})$$