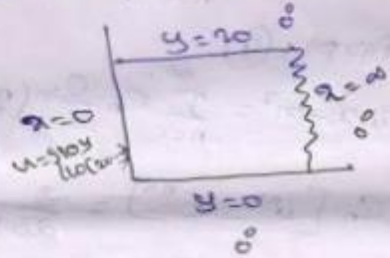
TOPIC 11:FOURIER SERIES SOLUTION IN CARTESIAN CO-ORDINATES

A rectangular plate with insulated surfaces is 20cm wide and so long compared to its width that it may be considered infinite in length. If the temp. at the short edge $x=0$ is

$$u = \begin{cases} 10y & 0 \le y \le 10 \\ 10(20-y) & 10 \le y \le 10. \end{cases}$$

and the two long edges as well as the other short edge are kept at $0°C$ find the steady state temp. distribution in the plate.



Sol: The temp $u(x,y)$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

The cond. are.

(1) $u(x,0) = 0$

(2) $u(x,20) = 0$

(3) $u(\infty,y) = 0$

(4) $u(0,y) = u = \begin{cases} 10y & 0 \le y \le 10 \\ 10(20-y) & 10 \le y \le 20. \end{cases}$

The Suitable Sol. is

$$u(x,y) = (Ae^{px} + Be^{-px})(C\cos py + D\sin py) \quad \text{--- (I)}$$

Apply (1) :

$$0 = (Ae^{px} + Be^{-px})\,C$$

$$C = 0$$

Apply (2) :

$$0 = (Ae^{px} + Be^{-px})(D\sin 20p)$$

$$\sin 20p = 0 = \sin n\pi$$

$$p = n\pi/20$$

Apply (3) $x = \infty$

$$e^{-\infty} = 0 \qquad e^{\infty} = \infty$$

$$0 = [Ae^{\infty} + Be^{-\infty}]\,D\sin py$$

$$= (A\infty)(D\sin py) = 0$$

$$\boxed{A = 0}$$

$\therefore$ (I) Becomes

$$u(x,y) = Be^{-\frac{n\pi x}{20}} D\sin\left(\frac{n\pi y}{20}\right)$$

Most gen. sol.

$$u(x,y) = \sum_{n=1}^{\infty} D_n e^{-\frac{n\pi x}{20}} \sin\left(\frac{n\pi y}{20}\right)$$

Apply (4)

$$u = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi y}{20}\right)$$

$$D_n = \frac{2}{20}\int_0^{20} u \sin\left(\frac{n\pi y}{20}\right) dy$$

$$= \frac{1}{10}\left[\int_0^{10} 10y \sin\left(\frac{n\pi y}{20}\right) dy + \int_{10}^{20} 10(20-y)\sin\left(\frac{n\pi y}{20}\right) dy\right]$$

$$= \left[y\left(-\cos\left(\frac{n\pi y}{20}\right)\left(\frac{20}{n\pi}\right)\right) + \sin\left(\frac{n\pi y}{20}\right)\left(\frac{400}{n^2\pi^2}\right)\right]_0^{10} +$$

$$\left[(20-y)\left\{-\cos\left(\frac{n\pi y}{20}\right)\left(\frac{20}{n\pi}\right) - \sin\left(\frac{n\pi y}{20}\right)\left(\frac{400}{n^2\pi^2}\right)\right\}\right]$$

$$= \left[\frac{-200}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{400}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) + \frac{200}{n\pi} \cos\left(\frac{n\pi}{2}\right) + \frac{400}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \right]$$

$$D_n = \frac{800}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right)$$

$$u(x,y) = \sum_{n=1}^{\infty} \frac{800}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) e^{-\frac{n\pi x}{20}} \sin\left(\frac{n\pi y}{20}\right)$$