



TOPIC 7: Convolution Theorem



Convolution Theorem

Definition :: (convolution of two sequences)

i, The convolution of two sequences $\{f(n)\}$ & $\{g(n)\}$ is defined as,

$$i) \{f(n) * g(n)\} = \sum_{k=-\infty}^{\infty} f(k) g(n-k)$$

if the sequences are non casual

$$ii, \{f(n) * g(n)\} = \sum_{k=0}^{n} f(k) g(n-k)$$

if the sequences are casual.

ii, The convolution of two functions $f(t)$ & $g(t)$ is defined as,

$$\{f(t) * g(t)\} = \sum_{k=0}^n f(kT) g(n-k)T$$

where T is the sampling period.



Convolution Theorem on Z-Transform

Statement ::

$$i) Z \{f(n) * g(n)\} = F[z] \cdot G_1[z]$$

where,

$$Z \{f(n)\} = F[z] \quad \& \quad Z \{g(n)\} = G_1[z]$$



TOPIC 7: Convolution Theorem

$$ii) \mathcal{Z}[f(t) * g(t)] = F(z) \cdot G(z)$$

where,

$$\mathcal{Z}[f(t)] = F(z) \quad \& \quad \mathcal{Z}[g(t)] = G(z)$$

NOTE:

$$\mathcal{Z}^{-1}[F(z)G(z)] = \mathcal{Z}^{-1}[F(z)] * \mathcal{Z}^{-1}[G(z)]$$

1) Find $\mathcal{Z}^{-1}\left[\frac{z^2}{z^2 - a^2}\right]$

Solution:

$$\mathcal{Z}^{-1}\left[\frac{z^2}{z^2 - a^2}\right] = \mathcal{Z}^{-1}\left[\frac{z}{z-a}\right] * \frac{z}{z-a}$$

$$= \mathcal{Z}^{-1}\left[\frac{z}{z-a}\right] * \mathcal{Z}^{-1}\left[\frac{z}{z-a}\right]$$

$$= \sum_{k=0}^{\infty} a^k * a^{n-k}$$

$$= \sum_{k=0}^n a^k a^{n-k}$$

$$= a^n \sum_{k=0}^n \left(\frac{a^k}{a^k}\right)$$

$$= a^n \sum_{k=0}^n \left(\frac{a}{a}\right)^k$$

$$= a^n \sum_{k=0}^n (1)^k$$



TOPIC 7: Convolution Theorem

$= a^n (n+1)$

2, Find $Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right]$

Solution:

$$Z^{-1} \left[\frac{z^2}{(z-a)(z-b)} \right] = Z^{-1} \left[\frac{z}{z-a} \cdot \frac{z}{z-b} \right]$$

$$= Z^{-1} \left[\frac{z}{z-a} \right] * Z^{-1} \left[\frac{z}{z-b} \right]$$

$$= a^n * b^n$$

$$= \sum_{k=0}^n a^k b^{n-k}$$

$$= \sum_{k=0}^n a^k b^{n-k}$$

$$= b^n \sum_{k=0}^n \left(\frac{a}{b}\right)^k$$

NOTE:

In a G.P. $1 + r + r^2 + \dots + r^n = \frac{r^{n+1} - 1}{r - 1}, r > 1$

$1 + r + r^2 + \dots + r^n = \frac{1 - r^{n+1}}{1 - r}, r < 1$

if $r > 1$

$$Z^{-1} \frac{z^2}{(z-a)(z-b)} = b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{\frac{a}{b} - 1} \right]$$



TOPIC 7: Convolution Theorem

$$z^{-1} \frac{z^n}{(z-a)(z-b)} = \left[\begin{matrix} n \\ b \end{matrix} \right] \frac{z^{n-1} - (a/b)^{n+1}}{1 - a/b}$$

3. Find the inverse z-transform of $\frac{8z^2}{(2z-1)(4z-1)}$ using Convolution theorem.

Solution :-

$$\frac{8z^2}{(2z-1)(4z-1)} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$z^{-1} \left[\frac{8z^2}{(2z-1)(4z-1)} \right] = z^{-1} \left[\frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})} \right]$$

$$= z^{-1} \left[\frac{z}{z-\frac{1}{2}} \cdot \frac{z}{z-\frac{1}{4}} \right]$$

$$= z^{-1} \left[\frac{z}{z-\frac{1}{2}} \right] * z^{-1} \left[\frac{z}{z-\frac{1}{4}} \right]$$

$$= \left(\frac{1}{2} \right)^n * \left(\frac{1}{4} \right)^n$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^k \left(\frac{1}{4} \right)^{n-k}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^n \sum_{k=0}^n \left(\frac{1}{4} \right)^k$$

$$= \left(\frac{1}{4} \right)^n \sum_{k=0}^n \left(\frac{1}{2} \right)^k$$

$$= \left(\frac{1}{4} \right)^n \sum_{n=0}^{\infty} \left(\frac{1}{2} \right)^k$$



TOPIC 7: Convolution Theorem

The image shows a handwritten derivation on a grid background. It starts with the inverse Z-transform of a rational function:

$$Z^{-1} \left[\frac{8z^2}{(2z-1)(4z-1)} \right] = \left(\frac{1}{4}\right)^n \frac{2^{n+1} - 1}{2 - 1}$$

Below this, there is a line with some faint, illegible text and another equation:

$$= \left(\frac{1}{4}\right)^n 2^{n+1} - 1$$

At the bottom, there is a final line with the same expression as the first equation, but with a different denominator:

$$\therefore Z^{-1} \left[\frac{8z^2}{(2z-1)(4z-1)} \right] = \left(\frac{1}{4}\right)^n \frac{2^{n+1}}{1 - 2^{-1}}$$