TOPIC 7: Convolution Theorem

Convolution Theorem

Definition: (convolution of two sequences)

i) The convolution of two sequences $\{f(n)\}$ & $\{g(n)\}$ is defined as,

i) $$\{f(n) * g(n)\} = \sum_{K=-\infty}^{\infty} f(K)\, g(n-K)$$

if the sequences are non casual

ii) $$\{f(n) * g(n)\} = \sum_{K=0}^{n} f(K)\, g(n-K)$$

if the sequences are casual.

ii) The convolution of two functions $f(t)$ & $g(t)$ is defined as,

$$\{f(t) * g(t)\} = \sum_{k=0}^{n} f(kT)\, g(n-k)T$$

where T is the sampling period.

Convolution Theorem on Z-Transform

Statement:

i) $Z\{f(n) * g(n)\} = F[z] \cdot G[z]$

where,

$Z[f(n)] = F[z]$ & $Z[g(n)] = G[z]$

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ii) $Z[f(t) * g(t)] = F[z] \cdot G[z]$

where,
$Z[f(t)] = F[z]$ & $Z[g(t)] = G[z]$

NOTE:

$$z^{-1}[F(z)\,G(z)] = z^{-1}[F(z)] * z^{-1}[G(z)]$$

1. Find $z^{-1}\left[\frac{z^2}{z^2-a^2}\right]$

Solution:

$$z^{-1}\left[\frac{z^2}{z^2-a^2}\right] = z^{-1}\left[\frac{z}{(z-a)} \cdot \frac{z}{(z-a)}\right]$$

$$= z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-a}\right]$$

$$= a^n * a^n$$

$$= \sum_{k=0}^{n} a^k a^{n-k}$$

$$= \sum_{k=0}^{n} a^k a^n a^{-k}$$

$$= a^n \sum_{k=0}^{\infty} \left(\frac{a^k}{a^k}\right)$$

$$= a^n \sum_{k=0}^{\infty} \left(\frac{a}{a}\right)^k$$

$$= a^n \sum_{k=0}^{\infty} (1)^k$$

$a^k a^{n-k}$

$a^n a^{-k}$

$\frac{a^n}{a^k}$

$a^n \cdot \left(\frac{a}{a}\right)^k$

$a^n (1)^k$

(n+1) times]

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$$= a^n (n+1)$$

2. Find $Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right]$

Solution:

$$Z^{-1}\left[\frac{z^2}{(z-a)(z-b)}\right] = Z^{-1}\left[\frac{z}{z-a}\cdot\frac{z}{z-b}\right]$$

$$= Z^{-1}\left[\frac{z}{z-a}\right] * Z^{-1}\left[\frac{z}{z-b}\right]$$

$$= a^n * b^n$$

$$= \sum_{k=0}^{n} a^k b^{n-k}$$

$$= \sum_{k=0}^{n} a^k b^n b^{-k}$$

$$= b^n \sum_{k=0}^{n} \left(\frac{a}{b}\right)^k$$

NOTE:

In a G.P. $1+r+r^2+\dots r^n = \frac{r^{n+1}-1}{r-1}, r>1$

$1+r+r^2+\dots r^n = \frac{1-r^{n+1}}{1-r}, r<1$

if $r>1$

$$Z^{-1}\frac{z^2}{(z-a)(z-b)} = b^n\left[\frac{\left(\frac{a}{b}\right)^{n+1}-1}{\frac{a}{b}-1}\right]$$

if $r < 1$

$$z^{-1}\frac{z^2}{(z-a)(z-b)} = \left[b^n\left[\frac{1-\left(\frac{a}{b}\right)^{n+1}}{1-\frac{a}{b}}\right]\right]$$

3. Find the inverse z-transform of $\frac{8z^2}{(2z-1)(4z-1)}$ by using convolution theorem

Solution :-

$$\frac{8z^2}{(2z-1)(4z-1)} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{4})}$$

$$z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] = z^{-1}\frac{z^2}{[z-\frac{1}{2}][z-\frac{1}{4}]}$$

$$= z^{-1}\left[\frac{z}{z-\frac{1}{2}}\cdot\frac{z}{z-\frac{1}{4}}\right]$$

$$= z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] * z^{-1}\left[\frac{z}{z-\frac{1}{4}}\right]$$

$$= \left(\tfrac{1}{2}\right)^n * \left(\tfrac{1}{4}\right)^n$$

$$= \sum_{n=0}^{k}\left(\tfrac{1}{2}\right)^k\left(\tfrac{1}{4}\right)^{n-k}$$

$$= \sum_{n=0}^{k}\left(\tfrac{1}{2}\right)^k\left(\tfrac{1}{4}\right)^n\left(\tfrac{1}{4}\right)^k$$

$$= \left(\tfrac{1}{4}\right)^n\sum_{n=0}^{k}\left(\frac{\frac{1}{2}}{\frac{1}{4}}\right)^k$$

$$= \left(\tfrac{1}{4}\right)^n\sum_{n=0}^{k} 2^k$$

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$$Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] = \left(\frac{1}{4}\right)^n \frac{2^{n+1}-1}{2-1}$$

$$= \left(\frac{1}{4}\right)^n \left(2^{n+1}-1\right)$$

$$\therefore Z^{-1}\left[\frac{8z^2}{(2z-1)(4z-1)}\right] = \left(\frac{1}{4}\right)^n \left(2^{n+1}-1\right)$$