TOPIC 9: Initial Value and Final value Theorem

Initial Value Theorem

If $Z[f(t)] = F[z]$ then

$$f(0) = \lim_{z \to \infty} F(z)$$

Proof:

Given, $F(z) = Z\, f(t) = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$F(z) = f(0.T) z^{-0} + f(1.T) z^{-1} + f(2.T) z^{-2} + \cdots$$

$$= f(0) + \frac{f(1.T)}{z} + \frac{f(2.T)}{z^2} + \cdots$$

$$\lim_{z \to \infty} F[z] = \lim_{z \to \infty} \left[f(0) + \frac{f(1.T)}{z} + \frac{f(2.T)}{z^2} + \cdots \right]$$

$$\lim_{z \to \infty} F[z] = f(0)$$

Final Value Theorem

If $Z[f(t)] = F(z)$,

$$\lim_{t \to \infty} f(t) = \lim_{z \to 1} (z-1) F(z)$$

Proof:

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$$Z\left[f(t+T)-f(t)\right]=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]z^{-n}$$

$$Z\left[f(t+T)\right]-Z\left[f(t)\right]=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]z^{-n}$$

$$zF[z]-zf(0)-F(z)=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]z^{-n}$$

$$(z-1)F[z]-zf(0)=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]z^{-n}$$

$$\lim_{z\to 1}(z-1)F(z)-zf(0)=\lim_{z\to 1}\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]z^{-n}$$

$$=\cancel{\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]}$$

$$=\cancel{\lim_{n\to\infty}\left[f(T)-f(0)+f(2T)-f(T)+\dots\dots f(n+1)T-f(nT)\right]}$$

$$=\cancel{\lim_{n\to\infty}}$$

$$\lim_{z\to 1}(z-1)F(z)-\lim_{z\to 1}zf(0)=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right](1)^{-n}$$

$$\lim_{z\to 1}(z-1)F(z)-f(0)=\sum_{n=0}^{\infty}\left[f(nT+T)-f(nT)\right]$$

$$=\lim_{n\to\infty}\left[f(T)-f(0)+f(2T)-f(T)+\dots f(nT+T)-f(nT)\right]$$

$$=\lim_{n\to\infty}f((n+1)T)-f(0)$$

$$\lim_{z\to 1}(z-1)F(z)-\cancel{f(0)}=f(\infty)-\cancel{f(0)}$$

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$$\lim_{z \to 1} (z-1) F(z) = f(\infty)$$

$$\lim_{z \to 1} (z-1) F(z) = \lim_{t \to \infty} f(t)$$