**TOPIC 11: Solution of difference equation**

## Solution of difference equation using Z-transform

To slove difference equ we need following results,

$$Z[y(n+1)] = z F[z] - z\, y(0)$$

$$Z[y(n+2)] = z^2 F[z] - z^2 y(0) - z\, y(1)$$

$$Z[y(n+3)] = z^3 F[z] - z^3 y(0) - z^2 y(1) - z\, y(2)$$

where, $Z[y(n)] = F[z]$

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1, Solve : $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

Given :

$y_0 = y_1 = 0$

Solution :

$y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

$z[y_{n+2}] + 6\,z[y_{n+1}] + 9\,z[y_n] = z[2^n]$

$[z^2 F(z) - z^2 y(0) - z\,y(1)] + 6\,[z F[z] - z\,y[0]] + 9\,F[z] = \frac{z}{z-2}$

$[z^2 F(z) - 0 - 0] + 6\,[z F(z) - 0] + 9\,F(z) = \frac{z}{z-2}$

$z^2 F(z) + 6z F(z) + 9F[z] = \frac{z}{z-2}$

$[z^2 + 6z + 9]\,F[z] = \frac{z}{z-2}$

$(z+3)^2\,F(z) = \frac{z}{z-2}$

$F(z) = \frac{z}{(z-2)(z+3)^2}$

$\frac{F(z)}{z} = \frac{1}{(z-2)(z+3)^2} = \frac{A}{z-2} + \frac{B}{z+3} + \dots$

$\frac{1}{(z-2)(z+3)^2} = \frac{A(z+3)^2 + B(z+3)(z-2) + C(z-2)}{(z-2)(z+3)^2}$

$1 = A(z+3)^2 + B(z+3)(z-2) + C(z-2)$

put $z = -3$

$1 = 0 + 0 + C(-5)$

$1 = -5C$

$\boxed{C = -\frac{1}{5}}$

put $z = 2$

$1 = A(5)^2 + 0 + 0$

$1 = 25A$

$\boxed{A = \frac{1}{25}}$

Put $z = 0$

$1 = 9A + (-6)B + (-2)C$

$1 = 9 \times \frac{1}{25} - 6B - 2(-\frac{1}{5})$

$1 = \frac{9}{25} - 6B + \frac{10}{5}$

$1 = \frac{9}{25} + \frac{10}{25} - 6B \Rightarrow 1 = \frac{19}{25} - 6B$

$1 - \frac{19}{25} = -6B$

$\frac{6}{25} = -6B$

$\boxed{B = -\frac{1}{25}}$

$\cancel{= \frac{9 + 28}{25} - 6B}$

$\cancel{1 = \frac{129}{25} - 6B}$

$\cancel{1 - \frac{29}{25} = -6B}$

$\cancel{-\frac{4}{25} =}$

$$\frac{F(z)}{z} = \frac{\frac{1}{25}}{z-2} + \frac{-\frac{1}{25}}{z+3} + \frac{(-\frac{1}{25})}{(z+3)^2}$$

$$\frac{F(z)}{z} = \frac{1}{25}\frac{1}{z-2} - \frac{1}{25}\frac{1}{z+3} - \frac{1}{25}\frac{1}{(z+3)^2}$$

$$= \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{25}\frac{z}{(z+3)^2}$$

$$Z[y(n)] = \frac{1}{25}\frac{z}{z-2} - \frac{1}{25}\frac{z}{z+3} - \frac{1}{25 \cdot 3}\frac{3z}{(z+3)^2}$$

$$y(n) = \frac{1}{25} Z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} Z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{75} Z^{-1}\left[\frac{3z}{(z+3)^2}\right]$$

$$= \frac{1}{25}\left[2^n\right] - \frac{1}{25}\left[(-3)\right]^n + \frac{1}{15}\, n\,(-3)^n$$

$$= \frac{1}{25}\left[2^n - (-3)^n - \frac{5}{3}\, n\,(-3)^n\right]$$

2) $y(n+3) - 3y(n+1) + 2y(n) = 0$

$y(0) = 4,\ y(1) = 0 \ \&\ y(2) = 8$

Solution:

$$Z\left[y(n+3)\right] - 3Z\,y(n+1) + 2Z(y(n)) = 0$$

$$z^3 F[z] - z^3 y(0) - z^2 y(1) - z y(2) - 3\left[z F[z] - z y[0]\right] + 2\left[F[z]\right] = 0$$

$$\left[z^3 F[z] - 4z^3 - 0 - 8z\right] - 3\left[z F(z) - 4z\right] + 2F[z] = 0$$

$$z^3 F[z] - 4z^3 - 8z - 3z F[z] + 12z + 2F[z] = 0$$

$$z^3 F[z] - 3z F[z] + 2F[z] = 4z^3 + 8z - 12z$$

$$F[z]\left[z^3 - 3z + 2\right] = 4z^3 - 4z$$

$$F[z] = \frac{4z^3 - 4z}{z^3 - 3z + 2} = \frac{4z^3 - 4z}{(z-1)^2 (z+2)}$$

$$F[z] = \frac{4z^3 - 4z}{(z-1)^2 (z+3)}$$

$z^3 - 3z + 2$

| | | | |
|---|---|---|---|
| 1 | 0 | -3 | 2 |
| 0 | 1 | 1 | -2 |
| 1 | 1 | -2 | 0 |

$(x-1)(x^2+x-2) = 0$

$(x-1)(x-1)(x+2) = 0$

$(x-1)^2(x+2) = 0$

$$F[z]\, z^{n-1} = \frac{z\,(4z^2 - 4)\, z^{n-1}}{(z-1)^2 (z+3)}$$

$$F[z]\, z^{n-1} = \frac{\left[4z^2 - 4\right] z^n}{(z-1)^2 (z+3)}$$

$$= \frac{4\,(z+1)\cancel{(z-1)}\, z^n}{(z-1)^{\cancel{2}} (z+2)}$$

$$= 4\,(z+1)\, z^n$$

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$$F[z]\, z^{n-1} = \frac{4(z+1)\, z^n}{(z-1)(z+2)}$$

$z=1$ is a simple pole

$z=-2$ is a simple pole

$$\underset{\text{at } z=1}{\text{Res}} = \lt_{z\to 1} \cancel{(z-1)} \frac{4(z+1)\, z^n}{\cancel{(z-1)}(z+2)}$$

$$= \lt_{z\to 1} \frac{4(z+1)\, z^n}{z+2}$$

$$= 8/3$$

$$\underset{\text{at } z=-2}{\text{Res}} = \lt_{z\to -2} \cancel{(z+2)} \frac{4(z+1)\, z^n}{(z-1)\cancel{(z+2)}}$$

$$= \lt_{z\to -2} \frac{4(z+1)\, z^n}{(z-1)}$$

$$= \frac{-4(-2)^n}{-3} = 4/3\, (-2)^n$$

$y(n)$ = sum of the residues

$$= 8/3 + 4/3\, (-2)^n$$