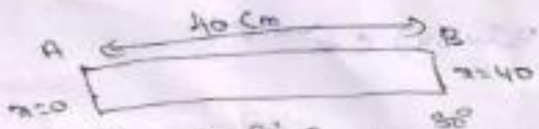
TOPIC 7: PROBLEMS ON ONE DIMENSIONAL HEAT EQUATION

The ends A and B of a rod 40cm long have their temp. kept at 0°C and 80°C resp. until steady state cond. prevails. The temp. of the end B is then suddenly reduced to 40°C and kept so. while that of the end A is kept at 0°C find the subsequent temp. distribution $u(x,t)$ in the rod.



0° $a = \frac{80-0}{40} = 2$ 80°

$b = 0$

0° $ax+b = 2x$ 40°

$\boxed{\frac{40-0}{40} = 1}$

$ax+b = x.$

The temp. $u(x,t)$ is from $\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$

The Cond. are

(1) $u(0,t) = 0$ (2) $u(40,t) = 40$ (3) $u(x,0) = ax+b = 2x.$

The suitable sol.

$$u(x,t) = (A\cos px + B\sin px)\, Ce^{-\alpha^2 p^2 t} + u_s(x)$$

$$u_s(x) = ax+b = x.$$

$$u(x,t) = x + (A\cos px + B\sin px)\, Ce^{-\alpha^2 p^2 t} \quad \text{— (1)}$$

Apply (1)

$A\,(Ce^{-\alpha^2 p^2 t}) = 0 \Rightarrow A = 0$

Apply ②

$$40 = 40 + (B \sin 40p)\, C e^{-\alpha^2 p^2 t}$$

$$(B \sin 40p)\, C e^{-\alpha^2 p^2 t} = 0$$

$$\sin 40p = \sin n\pi \qquad p = n\pi/40.$$

$$u(x,t) = x + B \sin\left(\frac{n\pi x}{40}\right) C e^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$

Most gen. sol. is

$$u(x,t) = x + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$

Apply (3) put $t=0$

$$2x = x + \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right)$$

$$x = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{40}\right)$$

$$C_n = \frac{2}{40} \int_0^{40} x \sin\left(\frac{n\pi x}{40}\right) dx.$$

$$= \frac{1}{20} \left[x\left(-\cos\left(\frac{n\pi x}{40}\right)\left(\frac{40}{n\pi}\right)\right) + \sin\left(\frac{n\pi x}{40}\right)\left(\frac{1600}{n^2\pi^2}\right)\right]_0^{40}$$

$$= \frac{1}{20}\left[-\frac{x\,40}{n\pi}\cos\left(\frac{n\pi x}{40}\right) + \frac{1600}{n^2\pi^2}\sin\left(\frac{n\pi x}{40}\right)\right]_0^{40}$$

$$= \frac{1}{20}\left[-\frac{1600}{n\pi}(-1)^n\right] = -\frac{80(-1)^n}{n\pi}$$

$$u(x,t) = x + \sum_{n=1}^{\infty} \frac{-80(-1)^n}{n\pi} \sin\left(\frac{n\pi x}{40}\right) e^{-\frac{\alpha^2 n^2 \pi^2 t}{1600}}$$