TOPIC 3: Elementary Properties of Z transforms

Linearity Property

The z-Transform is linear

i) $z\{af(n) + bg(n)\} = a\,z\{f(n)\} + b\,z\{g(n)\}$

ii) $z\{af(t) \pm bg(t)\} = a\,z\{f(t)\} \pm b\,z\{g(t)\}$

Proof i):

w.k.t. $z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$z\{af(n) + bg(n)\} = \sum_{n=0}^{\infty} \{af(n) + bg(n)\}\, z^{-n}$$

$$= a \sum_{n=0}^{\infty} f(n)\, z^{-n} + b \sum_{n=0}^{\infty} g(n)\, z^{-n}$$

Proof:

w.k.T. $Z[f(t)] = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$Z[a f(t) \pm b g(t)] = \sum_{n=0}^{\infty} [a f(nT) \pm b g(nT)] z^{-n}$$

$$= \sum_{n=0}^{\infty} a f(nT) z^{-n} \pm b g(nT) z^{-n}$$

$$= a \sum_{n=0}^{\infty} f(nT) z^{-n} \pm b \sum_{n=0}^{\infty} g(nT) z^{-n}$$

$$= a Z[f(t)] \pm b Z[g(t)]$$

First Shifting Theorem

i, If $Z\{f(t)\} = F(z)$, then

$$Z\{e^{-at} f(t)\} = F[ze^{aT}]$$

ii, If $Z\{f(t)\} = F(z)$ then,

$$Z\{e^{at} f(t)\} = F[ze^{-aT}]$$

iii, $Z\{f(t)\} = F(z)$ then, $Z\{a^n f(t)\} = F[z/a]$

iv, $Z\{f(n)\} = F(z)$ then, $Z\{a^n f(n)\} = F[z/a]$

Proof:

i, Given, $F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$Z\{e^{-at} f(t)\} = \sum_{n=0}^{\infty} e^{-anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (e^{aT})^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (ze^{at})^{-n}$$

$$= Z[f(t)]_{z \to ze^{at}}$$

$$= F[z]_{z \to ze^{at}}$$

$$\therefore Z\{e^{-at} f(t)\} = F[ze^{at}]$$

ii, Given, $F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$Z\{e^{at} f(t)\} = \sum_{n=0}^{\infty} e^{anT} f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (e^{-aT})^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) (ze^{-aT})^{-n}$$

$$= Z[f(t)]_{z \to ze^{-aT}}$$

$$= F(z)_{z \to ze^{-aT}}$$

$$= F[ze^{-aT}]$$

$$\therefore Z\{e^{at} f(t)\} = F[ze^{-aT}]$$

iii, Given, $F(z) = Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$

$$Z\{a^n f(t)\} = \sum_{n=0}^{\infty} a^n f(nT) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(nT) \frac{z^{-n}}{a^{-n}}$$

$$= \sum_{n=0}^{\infty} f(nT) (z/a)^{-n}$$

$$= Z\{f(t)\}_{z \to z/a}$$

$$= F[z]_{z \to z/a}$$

$$= F[z/a]$$

iv, Given,

$$F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$Z\{a^n f(n)\} = \sum_{n=0}^{\infty} a^n f(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n) \frac{z^{-n}}{a^{-n}}$$

$$= \sum_{n=0}^{\infty} f(n) \left(\frac{z}{a}\right)^{-n}$$

$$= Z\{f(n)\}_{z \to z/a}$$

$$= F[z]_{z \to z/a}$$

$$= F[z/a]$$

$$\therefore Z\{a^n f(n)\} = F[z/a]$$

Differentiation in the Z-Domain

i, $Z\{n f(t)\} = -z \frac{d}{dz} F(z)$

ii, $Z\{n f(n)\} = -z \frac{d}{dz} F(z)$

Proof:

i, $Z\{n f(t)\} = -z \frac{d}{dz} F(z)$

W.K.T. $F(z) = Z\{f(t)\}$

$$F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$$

$$\frac{d}{dz} F(z) = \sum_{n=0}^{\infty} f(nT)\, (-n) z^{-n-1}$$

$$= \sum^{\infty} -f(nT)\, n\, z^{-n} z^{-1}$$

$\frac{d}{dz} F(z) = -\sum_{n=0}^{\infty} f(nT)\; n \cdot \frac{z^{-n}}{z}$

$-z \frac{d}{dz} F(z) = \sum_{n=0}^{\infty} f(nT)\; n\, z^{-n}$

$-z \frac{d}{dz} F(z) = Z\{n\, f(t)\}$

$\therefore -z \frac{d}{dz} F(z) = Z\{n\, f(t)\}$

ii, $Z\{n\, f(n)\} = -z \frac{d}{dz} F(z)$

W.k.T.

$F(z) = Z\{f(t)\}$

$F(z) = \sum_{n=0}^{\infty} f(nT)\, z^{-n}$

$\frac{d}{dz} F(z) = \sum_{n=0}^{\infty} f(n) \cdot -n \cdot z^{-n} \cdot z^{-1}$

$= -\sum_{n=0}^{\infty} f(n) \cdot n \frac{z^{-n}}{z}$

$-z \frac{d}{dz} F(z) = \sum_{n=0}^{\infty} f(n) \cdot n z^{-n}$

$= Z\{n f(n)\}$

$\therefore -z \frac{d}{dz} F(z) = Z\{n\, f(n)\}$

Second Shifting Theorem

i, If $Z[f(t)] = F(z)$ then,

$Z[f(t+T)] = zF(z) - z f(0)$

Proof:

$$Z\{f(t)\} = \sum_{n=0}^{\infty} f(nT) z^{-n}$$

$$Z[f(t+T)] = \sum_{n=0}^{\infty} f(nT+T) z^{-n}$$

$$= \sum_{n=0}^{\infty} (f(n+1)T) z^{-n}$$

$$= \sum_{n=0}^{\infty} (f(n+1)T) z^{-n} z^{1} \cdot z^{-1}$$

$$= z\sum_{n=0}^{\infty} (f(n+1)T) z^{-(n+1)} \quad \boxed{\therefore m = n+1}$$

$$= z\sum_{m=1}^{\infty} f(mT) z^{-m}$$

$$= z\left[\sum_{n=0}^{\infty} f(mT) z^{-m} - f(0)\right]$$

$$= z[F(z) - f(0)]$$

$$= zF(z) - z f(0)$$

$$\therefore Z[f(t)] = zF(z) - z f(0)$$

ii, $Z\{f(n+1)\} = F(z) - z f(0)$

Proof:

w.k.T. $F(z) = Z\{f(n)\} = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z\{f(n+1)\} = \sum_{n=0}^{\infty} f(n+1) z^{-n}$$

$$= \sum_{n=0}^{\infty} f(n+1) z^{-n} \cdot z^{1} \cdot z^{-1}$$

$$= z \sum_{n=0}^{\infty} f(n+1) z^{-(n+1)}$$

(Take $m = n+1$)

$$= z \sum_{m=1}^{\infty} f(m) z^{-m}$$

$$= z \sum_{m=0}^{\infty} \left[f(m) z^{-m} - f(0)\right]$$

$$= z\left[F(z) - f(0)\right]$$

$$= z F(z) - z f(0)$$

NOTE 1:

$Z[f(t)] = F(z)$ then,

$$Z\, f(t+kT) = Z\left[f(n+k)T\right]$$

$$= z^{k}\left[f(z) - f(0T) - \frac{F(1.T)}{z} - \frac{F(2.T)}{z^{2}} + \dots + \frac{f(k-1)T}{z^{k-1}}\right]$$

NOTE 2:

$$Z[f(n+2)] = z^{2}\left[f(z) - f(0) - f(1) z^{-1}\right]$$