TOPIC 2: Problems based on Z transforms

Standard Formulae :

1. Prove that $Z[1] = \frac{z}{z-1}, \quad |z| > 1$

Solution :

W.K.T. $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z[1] = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= \left(\tfrac{1}{z}\right)^0 + \left(\tfrac{1}{z}\right)^1 + \left(\tfrac{1}{z}\right)^2 + \cdots$$

$$= 1 + \tfrac{1}{z} + \left(\tfrac{1}{z}\right)^2 + \cdots$$

$$= \left(1 - \tfrac{1}{z}\right)^{-1} \qquad \left(\because (1-x)^{-1} = 1 + x + x^2 + \cdots\right)$$

$$= \left(\frac{z-1}{z}\right)^{-1}$$

$$= \frac{z}{z-1}$$

$\therefore \boxed{Z[1] = \frac{z}{z-1}}$ hence it is proved//.

2. Prove that $Z[a^n] = \frac{z}{z-a}, \ |z| > |$

Solution:-

w.k.T. $\quad Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \frac{1}{z^n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n$$

$$= \left(\frac{a}{z}\right)^0 + \left(\frac{a}{z}\right)^1 + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= 1 + \frac{a}{z} + \left(\frac{a}{z}\right)^2 + \cdots$$

$$= \left(1 - \frac{a}{z}\right)^{-1}$$

$$= \left(z-a\right)^{-1}$$

3. Prove that $Z(n) = \frac{z}{(z-1)^2}$, $|z| > 1$

Solution:

W.K.T. $Z\{f(n)\} = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$Z(n) = \sum_{n=0}^{\infty} n\, z^{-n}$$

$$= \sum_{n=0}^{\infty} n\, \frac{1}{z^n}$$

$$= \sum_{n=0} n\, (1/z)^n$$

$$= 0\cdot(1/z)^0 + 1\cdot(1/z)^1 + 2\cdot(1/z)^2 + \cdots$$

$$= 0 + (1/z) + 2(1/z)^2 + 3(1/z)^3 + \cdots$$

$$= (1/z) + 2(1/z)^2 + 3(1/z)^3 + \cdots$$

$$= (1/z)\left[1 + 2(1/z) + 3(1/z)^2 + \cdots\right]$$

$$= 1/z\,(1 - 1/z)^{-2} \quad (\because (1-x)^{-2} = 1 + 2x + 3x^2 + \cdots)$$

$$= 1/z \left(\frac{z-1}{z}\right)^{-2}$$

$$= 1/z \left(\frac{z}{z-1}\right)^{2}$$

$$= 1/z\, \frac{z^2}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$

$$\therefore \boxed{Z(n) = \frac{z}{(z-1)^2}}$$

4, Prove that $z\left[\frac{1}{n}\right] = \log\left(\frac{z}{z-1}\right)$ if $|z|>1$, $n>0$

Solution:

W.K.T. $z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$z\left[\frac{1}{n}\right] = \sum_{n=1}^{\infty} \frac{1}{n}\, z^{-n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \cdot \frac{1}{z^n}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{z}\right)^n$$

$$= \frac{1}{1}\left(\frac{1}{z}\right) + \frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \cdots$$

$$= \frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots$$

$\therefore \log(1-x)^{-1} = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$ $\quad = \log\left(1-\frac{1}{z}\right)^{-1}$

$$= +\log\left(\frac{z-1}{z}\right)^{-1}$$

$$= \log \frac{z}{z-1}$$

$$\therefore z\left(\frac{1}{n}\right) = \log\left(\frac{z}{z-1}\right)$$

5, Prove that $z\left[\frac{1}{n+1}\right] = z \log \frac{z}{z-1}$

Solution:

W.K.T. $z[f(n)] = \sum_{n=0}^{\infty} f(n)\, z^{-n}$

$$z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right) z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{n+1}\right)\left(\frac{1}{z}\right)^n$$

$$= 1 + \frac{1}{2}\left(\frac{1}{z}\right) + \frac{1}{3}\left(\frac{1}{z}\right)^2 + \cdots$$

Multiply by $z, \frac{1}{z}$

$$= z\left(\frac{1}{z}\right)\left[1 + \frac{\left(\frac{1}{z}\right)}{2} + \frac{\left(\frac{1}{z}\right)^2}{3} + \cdots\right]$$

$$= z\left[\frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots\right]$$

$$= z\left[\log\left(1 - \frac{1}{z}\right)^{-1}\right]$$

$$= z \log\left(\frac{z-1}{z}\right)^{-1}$$

$$= z \log\left(\frac{z}{z-1}\right)$$

$$\therefore Z\left[\frac{1}{n+1}\right] = z \log\left[\frac{z}{z-1}\right]$$

6. Prove that $Z\left[\frac{1}{n-1}\right] = \frac{1}{z} \log\left[\frac{z}{z-1}\right]$

$n > 1$

Solution:

W.K.T. $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$$Z\left[\frac{1}{n-1}\right] = \sum_{n=2}^{\infty}\left[\frac{1}{n-1}\right] z^{-n}$$

$$= \sum_{n=2}^{\infty}\left(\frac{1}{n-1}\right)\left(\frac{1}{z}\right)^n$$

$$= \frac{1}{1}\left(\frac{1}{z}\right)^2 + \frac{1}{2}\left(\frac{1}{z}\right)^3 + \frac{1}{3}\left(\frac{1}{z}\right)^4 + \cdots$$

$$= \left(\frac{1}{z}\right)^2 + \frac{\left(\frac{1}{z}\right)^3}{2} + \frac{\left(\frac{1}{z}\right)^4}{3} + \cdots$$

$$= \frac{1}{z}\left(\frac{1}{z} + \frac{\left(\frac{1}{z}\right)^2}{2} + \frac{\left(\frac{1}{z}\right)^3}{3} + \cdots\right)$$

$= \frac{1}{z}\left[\log\left(\frac{z-1}{z}\right)^{-1}\right]$

$= \frac{1}{z}\log\left(\frac{z}{z-1}\right)$

$\therefore Z\left[\frac{1}{n-1}\right] = \frac{1}{z}\log\left(\frac{z}{z-1}\right)$

7, Prove that $Z\left(\frac{1}{n!}\right) = e^{1/z}$

Solution:

W.K.T. $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$Z\left(\frac{1}{n!}\right) = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$

$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{z}\right)^n$

$= \frac{1}{0!}\left(\frac{1}{z}\right)^0 + \frac{1}{1!}\left(\frac{1}{z}\right)^1 + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \cdots$

$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots$

$= 1 + \frac{(1/z)}{1!} + \frac{(1/z)^2}{2!} + \cdots$

$\therefore Z\left(\frac{1}{n!}\right) = e^{1/z}$

8, ~~Prove~~ Find that $Z\left[\frac{1}{(n+1)!}\right]$

Solution:

W.K.T. $Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$

$Z\left(\frac{1}{(n+1)!}\right) = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} z^{-n}$

$= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\frac{1}{z}\right)^n$

$$= \frac{1}{1!}\left(\frac{1}{z}\right)^0 + \frac{1}{2!}\left(\frac{1}{z}\right)^1 + \frac{1}{3!}\left(\frac{1}{z}\right)^2 + \cdots$$

$$= 1 + \frac{(1/z)}{2!} + \frac{(1/z)^2}{3!} + \cdots$$

Multiply & Divide by z

$$= z\left(\frac{1}{z}\right)\left[\frac{1}{1!} + \frac{(1/z)}{2!} + \frac{(1/z)^2}{3!} + \cdots\right]$$

$$= z\left[\frac{1/z}{1!} + \frac{(1/z)^2}{2!} + \frac{(1/z)^3}{3!} + \cdots\right]$$

$$= z\left[e^{1/z} - 1\right]$$

$$= ze^{+1/z} - z$$

$$\therefore Z\left[\frac{1}{(n+1)!}\right] = ze^{+1/z} - z$$

7. Find $Z\left[\frac{1}{n(n+1)}\right]$

Solution:

$$\frac{1}{n(n+1)} = \frac{A}{n} + \frac{B}{n+1}$$

$$\frac{1}{n(n+1)} = \frac{A(n+1) + Bn}{n(n+1)}$$

$$A(n+1) + Bn = 1$$

Put $n = -1$

$$1 = 0 - B$$

$$\boxed{B = -1}$$

put $n=0$.

$1 = A + 0$

$\boxed{A=1}$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

$$Z\left[\frac{1}{n(n+1)}\right] = Z\left[\frac{1}{n}\right] - Z\left[\frac{1}{n+1}\right]$$

$$= \log\left[\frac{z}{z-1}\right] - z\log\left[\frac{z}{z-1}\right]$$

$$= \log \frac{z}{z-1}(1-z)$$

2, Find $Z\left[\frac{1}{(n+1)(n+2)}\right]$

Solution:

$$\frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$\frac{1}{\cancel{(n+1)(n+2)}} = \frac{A(n+2)+B(n+1)}{\cancel{(n+1)(n+2)}}$$

$A(n+2) + B(n+1) = 1$

put $n=-1$

$\boxed{A=1}$

put $n=-2$

$\boxed{B=-1}$

$$\frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

$$Z\left[\frac{1}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] - Z\left[\frac{1}{n+2}\right]$$

$= z \log\left[\frac{z}{z-1}\right] - \left[\frac{1}{2} + \frac{1}{3}z^{-1} + \frac{1}{4}z^{-2} + \cdots\right]$

$= z \log\left[\frac{z}{z-1}\right] - \left[\frac{1}{2} + \frac{1}{3z} + \frac{1}{4z^2} + \cdots\right]$

$= z \log\left[\frac{z}{z-1}\right] - z^2\left[\frac{1}{2}\left(\frac{1}{z}\right)^2 + \frac{1}{3}\left(\frac{1}{z}\right)^3 + \frac{1}{4}\left(\frac{1}{z}\right)^4 + \cdots\right]$

$= z \log\left[\frac{z}{z-1}\right] - z^2\left[\frac{(1/z)^2}{2} + \frac{(1/z)^3}{3} + \frac{(1/z)^4}{4} + \cdots\right]$

$= z \log\left[\frac{z}{z-1}\right] - z^2\left[\log(1-1/z)^{-1} - 1/z\right]$

$= z \log\left[\frac{z}{z-1}\right] - z^2\left[\log\left(\frac{z-1}{z}\right)^{-1} - 1/z\right]$

$= z \log\left[\frac{z}{z-1}\right] - z^2\left[\log\left(\frac{z}{z-1}\right) - 1/z\right]$

$= z \log\left[\frac{z}{z-1}\right]\left[1-z\right] + z$

H.W.: Find the z-transform of $f(n) = \frac{2n+3}{(n+1)(n+2)}$

1) Find $z[a^n \cdot n]$

Solution:

W.k.T. $z[a^n f(n)] = F[z/a]$

$z[a^n \cdot n] = F[z]_{z \to z/a}$ $\quad \therefore F[z] = z[f(n)]$

$= (z[n])_{z \to z/a}$ $\quad \therefore z[n] = \frac{z}{(z-1)^2}$

$= \left[\frac{z}{(z-1)^2}\right]_{z \to z/a}$

$= \frac{z/a}{}$

$$= \frac{z/a}{\left(\frac{z-a}{a}\right)^2}$$

$$= \frac{z/a}{\frac{(z-a)^2}{a^2}} = \frac{z}{a} \times \frac{a^2}{(z-a)^2}$$

$$= \frac{za}{(z-a)^2}$$

$$\therefore Z\left[a^n \cdot n\right] = \frac{za}{(z-a)^2}$$

2) Find $Z\left[\frac{a^n}{n!}\right]$

Solution:

$$Z\left[\frac{a^n}{n!}\right] = Z\left[a^n \cdot \frac{1}{n!}\right]$$

w.k.T. $Z\left[a^n f(n)\right] = F\left[z/a\right]$

$$Z\left[a^n \cdot \frac{1}{n!}\right] = F[z]_{z \to z/a}$$

$$= \left[Z\left(\frac{1}{n!}\right)\right]_{z \to z/a}$$

$$= \left(e^{1/z}\right)_{z \to z/a}$$

$$= e^{\frac{1}{z/a}}$$

$$= e^{a/z}$$

$$\therefore Z\left[\frac{a^n}{n!}\right] = e^{a/z}$$

Find the Z-transform of $f(n) = \frac{2n+3}{(n+1)(n+2)}$

$Z\left[a^n \cos n\theta\right]$

$Z\left[a^n \cdot n\right]$

1, $Z\left[\frac{2n+3}{(n+1)(n+2)}\right]$

Solution:

$$\frac{2n+3}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$$

$$\frac{2n+3}{(n+1)\cancel{(n+2)}} = \frac{A(n+2)+B(n+1)}{(n+1)\cancel{(n+2)}}$$

But $n = -2$

$$2(-2)+3 = B(-2+1)$$

$$-1 = -B$$

$$\boxed{B = 1}$$

But $n = -1$

$$2(-1)+3 = A(-1+2)$$

$$\boxed{1 = A}$$

$$\frac{2n+3}{(n+1)(n+2)} = \frac{1}{n+1} + \frac{1}{n+2}$$

$$Z\left[\frac{2n+3}{(n+1)(n+2)}\right] = Z\left[\frac{1}{n+1}\right] + Z\left[\frac{1}{n+2}\right]$$

$$= \left[z \log \frac{z}{z-1}\right] + \left[z^2 \log \frac{z}{z-1}\right]$$

2, $Z\left[a^n \cos n\theta\right]$

$$Z\left[a^n f(n)\right] = F\left[\frac{z}{a}\right]$$

$$= Z\left[\cos n\theta\right]_{z \to z/a}$$

$$= Z\left[\frac{z-\cos\theta}{z^2-2z\cos\theta+1}\right]_{z \to z/a}$$

$$= \frac{\left[\frac{z}{a} - \cos\theta\right]}{\frac{z}{a} - 2\cos\theta + \frac{a}{z}}$$

i) Find $z\left[n^2\right]$

Solution:

w.k.T. $z\left[n f(n)\right] = -z \frac{d}{dz}\left[F(z)\right]$

$$z\left[n \cdot n\right] = -z \frac{d}{dz}\left[z\left[n\right]\right]$$

$$= -z \frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$\frac{d}{dz}\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$

$$= -z\left[\frac{(z-1)^2(1) - z\,2(z-1)}{(z-1)^4}\right]$$

$$= -z\left[\frac{\cancel{(z-1)}\ \cancel{(z-1)}\ (z-1-2z)}{(z-1)^{\cancel{4}\,3}}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$= -z\left[\frac{-1-z}{(z-1)^3}\right]$$

$$= z\left[\frac{z+1}{(z-1)^3}\right]$$

$$= \frac{z^2+z}{(z-1)^3}$$

$$\therefore\ z\left[n^2\right] = \frac{z^2+z}{(z-1)^3}$$

2, Find the z-transform of $(n+1)(n+2)$

Solution:

$$Z[(n+1)(n+2)] = Z[n^2+2n+n+2]$$

$$= Z[n^2+3n+2]$$

$$= Z[n^2] + 3Z[n] + 2Z[1]$$

$$= \frac{z(z+1)}{(z-1)^3} + 3\frac{z}{(z-1)^2} + 2\frac{z}{(z-1)}$$

$$= \frac{z(z+1) + 3z(z-1) + 2z(z-1)^2}{(z-1)^3}$$

$$= \frac{z^2+z+3z^2-3z+2z(z^2-2z+1)}{(z-1)^3}$$

$$= \frac{\cancel{z^2}+\cancel{z}+\cancel{3z^2}-\cancel{3z}+2z^3-\cancel{4z^2}+\cancel{2z}}{(z-1)^3}$$

$$= \frac{2z^3}{(z-1)^3}$$

$$\therefore Z[(n+1)(n+2)] = \frac{2z^3}{(z-1)^3}$$

3. If $F(z) = Z\left[\frac{z-\cos aT}{z^2-2z\cos aT+1}\right]$ find $f(0)$ & find $\lim_{t\to\infty} f(t)$

Solution:

w.k.T.

By Initial value theorem,

$$f(0) = \lim F(z)$$

$$= \lim_{z \to \infty} \frac{z(z - \cos aT)}{z^2 - 2z\cos aT + 1}$$

$$= \frac{\infty}{\infty}$$

Apply L - Hospital rule,

$$= \lim_{z \to \infty} \frac{z(1) + (z - \cos aT)(1)}{2z - 2\cos aT}$$

$$= \frac{\infty}{\infty}$$

Apply L-Hospital rule,

$$= \lim_{z \to \infty} \frac{1+1}{2} = \lim_{z \to \infty} \frac{2}{2} = 1$$

$$\therefore \boxed{f(0) = 1}$$

By Final value theorem,

$$\lim_{z \to 1} (z-1) F(z) = \lim_{t \to \infty} f(t)$$

$$= \lim_{z \to 1} (z-1) \frac{z(z - \cos aT)}{z^2 - 2z\cos aT + 1}$$

$$= \lim_{z \to 1} \frac{(z-1)\left[[z(1) + (z - \cos aT)(1)] + z[z - \cos aT](1)\right]}{2z - 2\cos aT}$$

$$= \frac{1 - \cos aT}{2 - 2\cos aT}$$

$$= \frac{1 - \cos aT}{2(1-\cos aT)} = \frac{1}{2}$$