TOPIC : 1.10 – Exponential & Normal Distribution

Exponential Distribution

A continuous random variable X is said to follow an exponential distribution with parameter $\lambda > 0$, if its pdf is given by

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

1. Find the moment generating function of the exponential distribution and hence find mean and variance.

The pdf of the exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

MGF $M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} dx = \lambda \int_0^{\infty} e^{-x(\lambda - t)} dx$$

$$= \lambda \left[\frac{e^{-(\lambda - t)x}}{-(\lambda - t)} \right]_0^{\infty} = \frac{\lambda}{t - \lambda} [0 - 1] = \frac{\lambda}{\lambda - t}$$

$$\text{Mean} = \left[\frac{d}{dt} \{ M_x(t) \} \right]_{t=0} = \left[\frac{d}{dt} \left\{ \frac{\lambda}{\lambda - t} \right\} \right]_{t=0}$$

$$= \lambda \left[\frac{-1}{(\lambda - t)^2} (-1) \right]_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$$

$$E[x^2] = \left[\frac{d^2}{dt^2} \{ M_x(t) \} \right]_{t=0} = \left[\frac{d}{dt} \left\{ \frac{\lambda}{(\lambda - t)^2} \right\} \right]_{t=0}$$

$$= \lambda \left[\frac{-2}{(\lambda - t)^3} (-1) \right]_{t=0} = \frac{2\lambda}{\lambda^3} = \frac{2}{\lambda^2}$$

$$\text{Var}(x) = E[x^2] - [E(x)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

ate and prove memoryless property for
ential distribution.

If the random variable x follows
ential distribution, then

$x > s+t \,/\, x > t] = P[x > s]$ for all $s, t > 0$

alled the memoryless property.

:

he pdf of x $\quad f(x) = \lambda e^{-\lambda x}, \; x \geq 0$.

$$[x > s+t \,/\, x > t] = \frac{P[x > s+t \cap x > t]}{P[x > t]}$$

$$= \frac{P[x > s+t]}{P[x > t]} \longrightarrow (1)$$

$$P[x > t] = \int_t f(x)\,dx = \int_t \lambda e \quad dx$$

$$= \lambda \left(\frac{e^{-\lambda x}}{-\lambda}\right)_t^{\infty} = -\left[0 - e^{-\lambda t}\right] = e^{-\lambda t}$$

$$P[x > s+t] = e^{-\lambda(s+t)}$$

sub. in (1),

$$P[x > s+t \,/\, x > t] = \frac{e^{-\lambda(s+t)}}{e^{-\lambda t}} = \frac{e^{-\lambda s} e^{-\lambda t}}{e^{-\lambda t}}$$

$$= e^{-\lambda s}$$

$$= P[x > s]$$

. The time (in hours) required to repair a ma... exponentially distributed with parameter $\lambda = \frac{1}{2}$. (i) What is the prob. that a repair takes atleast 10 h... given ... 9 h... (ii) the repair...

Let x denote the repair time ...

Given x is exponential with $\lambda = \frac{1}{2}$

$\therefore$ pdf of x $f(x) = \lambda e^{-\lambda x}, \quad x \geq 0$

$$= \frac{1}{2} e^{-x/2}, \quad x \geq 0$$

(i) $P[x > 10 / x > 9] = P[x > 9+1 / x > 9]$

$= P[x > 1]$ by memoryless property

$= e^{-\lambda(1)} = e^{-1/2}$ $\quad [P(x>t) = e^{-\lambda t}]$

$= 0.6065$

(ii) $P[x > 2] = e^{-\lambda(2)} = e^{-1/2(2)} = e^{-1} = 0.3679$

(4) If a continuous random variable X follows distribution in the interval $(0,2)$ & continuous rando variable Y follows exponential distribution with par α, find α such that $P[X<1] = P[Y<1]$.

Given X is uniformly distributed over $(0,2)$

$\therefore$ pdf of X $f(x) = \frac{1}{b-a} = \frac{1}{2}, \ 0<x<2$

and Y is exponentially distributed with parameter

$\therefore$ pdf of Y $f(y) = \alpha e^{-\alpha y}, \ y \geq 0$.

Given that $P(X<1) = P(Y<1)$

$$\Rightarrow \int_0^1 f(x)\,dx = \int_0^1 \alpha e^{-\alpha y}\,dy$$

$$\Rightarrow \frac{1}{2}(x)_0^1 = \alpha \left(\frac{e^{-\alpha y}}{-\alpha}\right)_0^1$$

$$\Rightarrow \frac{1}{2} = -\left(e^{-\alpha} - 1\right) \Rightarrow e^{-\alpha} - 1 = -\frac{1}{2}$$

$$\Rightarrow e^{-\alpha} = \frac{1}{2}$$

$$\Rightarrow e^{\alpha} = 2 \quad \Rightarrow \boxed{\alpha = \log_e 2}$$