TOPIC : 1.5 – Discrete and Continuous Random Variables

Random variable

A random variable is a rule that assigns a numerical values to each possible outcome of an experiment.

A real-valued function defined on the outcome of a probability experiment is called a random variable.

1. Discrete Random Variable
2. Continuous Random Variable

Discrete Random Variable

A Discrete random variable is a random variable whose possible values constitute finite set of values or countably infinite set of values.

Probability Mass Function (PMF)

Let X be a one dimensional discrete random variable which takes the values $x_1, x_2, \cdots x_n$. To each possible outcome 'x_i', we can associate a number p_i, ie) $P[X = x_i] = P_i$, called the probability of x_i.

The numbers P_i satisfies the following conditions (i) $p(x_i) \geq 0 \quad \forall i$

(ii) $\sum_{i=1}^{\infty} p(x_i) = 1$.

The function $p(x)$ satisfying the above two conditions is called the Probability Mass Function.

Cumulative distribution (or) Distribution function of X

The cumulative distribution function $F(x)$ of a discrete random variable X with probability distribution $P(x)$ is given by

$$F(x) = P(X \le x) = \sum_{t \le x} p(t)$$

$$x = -\infty, \ldots, -1, 0, 1, \ldots \infty.$$

Properties of distribution function

(i) $F(-\infty) = P(X \le -\infty) = 0$

(ii) $F(\infty) = P(X \le \infty) = 1$

(iii) $F(x_1) \leq F(x_2)$ if $x_1 \leq x_2$

(iv) $P(X > x) = 1 - P(X \leq x)$

(v) $P(X \leq x) = 1 - P(X > x)$

Expected value of a Discrete random variable X

Let X be a discrete random variable assuming values $x_1, x_2, \dots x_n$ with corresponding probabilities $P_1, P_2, \dots P_n$. Then

$$E[X] = \sum_{i=1}^{n} x_i \, p(x_i)$$

is called the Expected value of X (or) Mean of X

The Variance of a Discrete random variable X

It is defined by $\operatorname{Var}(X) = E[X^2] - [E(X)]^2$

① Find the expected value of the discrete random variable X with the pmf $p(x) = \begin{cases} 1/3, & x = 0 \\ 2/3, & x = 2 \end{cases}$

$$E(X) = \sum x \, p(x)$$

$$= (0)(1/3) + (2)(2/3)$$

$$= 4/3$$

③ A random variable X has the following probability function

x	0	1	2	3	4	5	6	7	8
$P(x)$	a	$3a$	$5a$	$7a$	$9a$	$11a$	$13a$	$15a$	$17a$

(i) Determine the value of 'a'

(ii) Find $P(x<3)$, $P(x \geq 3)$, $P(0<x<5)$

(iii) Find the distribution function of X.

(i) WKT $\sum p(x) = 1$

$\Rightarrow a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$\Rightarrow 81a = 1$

$\Rightarrow \boxed{a = \frac{1}{81}}$

(ii) $P(x<3) = P(x=0) + P(x=1) + P(x=2)$

$= a + 3a + 5a = 9a = \frac{9}{81}$

$P(x \geq 3) = 1 - P(x<3)$

$= 1 - \frac{9}{81} = \frac{72}{81}$

$P(0<x<5) = P(x=1) + P(x=2) + P(x=3) + P(x=4)$

$= 3a + 5a + 7a + 9a = 24a = \frac{24}{81}$

(iii) To find the distribution function of x.

$X = x$	$F(x) = P(X \le x)$
0	$F(0) = P(X \le 0) = \frac{1}{81}$
1	$F(1) = P(X \le 1) = 4a = \frac{4}{81}$
2	$F(2) = P(X \le 2) = 9a = \frac{9}{81}$
3	$F(3) = P(X \le 3) = 16a = \frac{16}{81}$
4	$F(4) = P(X \le 4) = 25a = \frac{25}{81}$
5	$F(5) = P(X \le 5) = \frac{36}{81}$
6	$F(6) = P(X \le 6) = \frac{49}{81}$
7	$F(7) = P(X \le 7) = 64a = \frac{64}{81}$
8	$F(8) = P(X \le 8) = 81a = 1$

(5) A random variable X has the following probability distribution

X	0	1	2	3	4	5	6	7
P(x)	0	K	2K	2K	3K	K^2	$2K^2$	$7K^2+K$

Find (i) the value of K (ii) $P[1.5 < x < 4.5 / x > 2]$ (iii) the smallest value of n for which $P[x \le n] > \frac{1}{2}$

(i) WKT $\sum p(x) = 1$

$\Rightarrow$ $K + 2K + 2K + 3K + K^2 + 2K^2 + 7K^2 + K = 1$

$\Rightarrow$ $9K + 10K^2 - 1 = 0$

$\Rightarrow$ $K = \dfrac{-9 \pm \sqrt{81+40}}{20} = \dfrac{-9 \pm \sqrt{121}}{20}$

$= \dfrac{-9+11}{20}, \dfrac{-9-11}{20} = \dfrac{1}{10}, -1$

$\boxed{K = \frac{1}{10}}$

(ii) $P[1.5 < x < 4.5 / x > 2]$

$= \dfrac{P[(1.5 < x < 4.5) \cap (x > 2)]}{P[x > 2]}$

$= \dfrac{P[2 < x < 4.5]}{P[x > 2]} \rightarrow (1)$

$P[2 < x < 4.5] = P[x = 3] + P[x = 4]$

$= 2K + 3K = \frac{5}{10} = \frac{1}{2}$

$P[x > 2] = 1 - P[x \le 2]$

$= 1 - \{K + 2K\} = 1 - 3K = 1 - \frac{3}{10} = \frac{7}{10}$

Sub. in (1)

$P[1.5 < x < 4.5 / x > 2] = \dfrac{5/10}{7/10} = \dfrac{5}{7}$

(iii)

$X = x$	$F(x) = P(X \le x)$
0	$F(0) = P(X \le 0) = 0$
1	$F(1) = P(X \le 1) = k = 1/10$
2	$F(2) = P(X \le 2) = 3k = \frac{3}{10}$
3	$F(3) = P(X \le 3) = 5k = \frac{1}{2}$
4	$F(4) = P(X \le 4) = 8k = \frac{4}{5}$
5	$F(5) = P(X \le 5) = 8k + k^2 = \frac{8}{10} + \frac{1}{100} = \frac{81}{100}$
6	$F(6) = P(X \le 6) = 8k + 3k^2 = \frac{8}{10} + \frac{3}{100} = \frac{83}{100}$
7	$F(7) = P(X \le 7) = 1$

The smallest value of n for which

$$P[X \le n] > 1/2 \quad \text{is} \quad \boxed{n = 4}$$

9) Given the following probability distribution of x compute (i) $E[x]$ (ii) $E[x^2]$ (iii) $E[2x \pm 3]$ (iv) $Var[2x \pm 3]$

x	-3	-2	-1	0	1	2	3
p(x)	0.05	0.1	0.3	0	0.3	0.15	0.1

(i) $E[x] = \sum x\, p(x)$

$= (-3)(0.05) + (-2)(0.1) + (-1)(0.3) + (1)(0.3) + 2(0.15) + 3(0.1)$

$= 0.25$

(ii) $E(x^2) = \sum x^2 p(x)$

$= (9)(0.05) + (4)(0.1) + (1)(0.3) + (1)(0.3) + (4)(0.15) + (9)(0.1)$

$= 2.95$

(iii) $E[2x \pm 3] = 2E[x] \pm 3$

$= 2(0.25) \pm 3$

$= 0.5 \pm 3$

(iv) $Var(x) = E[x^2] - [E(x)]^2$

$= 2.95 - (0.25)^2$

$= 2.95 - 0.0625$

$= 2.8875$

$Var(2x \pm 3) = 4\, Var(x)$

$= 11.55$

Continuous Random Variable

A random variable X which takes all possible values in a given interval is called a continuous Random variable.

Probability Density Function

For a continuous random variable X, a [probability] density function is a function such that

(i) $f(x) \geq 0$

(ii) $\int_{-\infty}^{\infty} f(x)\, dx = 1$

(iii) $P(a \leq X \leq b) = \int_{a}^{b} f(x)\, dx$

Cumulative Distribution Function

If $f(x)$ is a pdf of a continuous random variable X, then the function

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\, dx \quad , \; -\infty < x < \infty$$

is called the cumulative distribution function of the random variable X.

Formula

(i) $f(x) = \frac{d}{dx}\left[F(x)\right]$

(ii) Mean $= E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$

(iii) $E[X^2] = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

(iv) $P[a \leq X \leq b] = F(b) - F(a)$

(v) $P[a \le X \le b] = P[a \le X < b] = P[a < X \le b]$
$= P[a < X < b]$, X being a continuous random variable.

① If the pdf of a random variable X is $f(x) = \frac{x}{2}$ in $0 \le x \le 2$, find $P[X > 1.5 / X > 1]$.

$$P[X > 1.5 / X > 1] = \frac{P[(X > 1.5) \cap (X > 1)]}{P[X > 1]}$$

$$= \frac{P[X > 1.5]}{P[X > 1]} \rightarrow (1)$$

$$P[X > 1.5] = \int_{1.5}^{2} f(x)\,dx = \int_{1.5}^{2} \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\right]_{1.5}^{2} = \frac{1}{4}\left[4 - 2.25\right]$$

$$= \frac{1}{4}[1.75] = 0.4375$$

$$P[X > 1] = \int_{1}^{2} f(x)\,dx = \int_{1}^{2} \frac{x}{2}\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_{1}^{2}$$

$$= \frac{1}{4}[4 - 1] = 0.75$$

sub in (1),

$$P[X > 1.5 / X > 1] = \frac{0.4375}{0.75} = 0.5833$$

2) Show that the function $f(x) = e^{-x}, x \geq 0$ is a probability density function of a random variable of x

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{0}^{\infty} e^{-x}\,dx = \left[\frac{e^{-x}}{-1}\right]_0^{\infty}$$

$$= -[0-1] = 1$$

$\therefore$ Given $f(x)$ is a pdf.

3) If $f(x) = \begin{cases} Ke^{-x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$ is the pdf of a random variable X, then find the value of K.

WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\Rightarrow \int_0^{\infty} Ke^{-x}\,dx = 1 \Rightarrow K\left[\frac{e^{-x}}{-1}\right]_0^{\infty} = 1$$

$$\Rightarrow -K[0-1] = 1 \Rightarrow \boxed{K = 1}$$

4) Assume that X is a continuous random variable with pdf $f(x) = \begin{cases} \frac{3}{4}(2x - x^2), & 0 \leq x \leq 2 \\ 0, & \text{elsewhere} \end{cases}$ Find $P[X>1]$

$$P[X>1] = \int_1^2 f(x)\,dx = \int_1^2 \frac{3}{4}(2x - x^2)\,dx$$

$$= \frac{3}{4}\left[\frac{\cancel{2}x^2}{\cancel{2}} - \frac{x^3}{3}\right]_1^2 = \frac{3}{4}\left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right]$$

$$= \frac{3}{4}\left[\frac{4}{3} - \frac{2}{3}\right] = \frac{3}{4}\left[\frac{2}{3}\right] = \frac{1}{2}$$

(15) The pdf of a random variable X is given by

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ k(2-x), & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

. Find

(i) the value of k

(ii) $P[0.2 < X < 1.2]$

(iii) $P[0.5 < X < 1.5 / X \geq 1]$

(iv) the distribution function of X.

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^1 x\,dx + \int_1^2 k(2-x)\,dx = 1$$

$$\Rightarrow \frac{1}{2} + k\left[2 - \frac{3}{2}\right] = 1$$

$$\Rightarrow k\left[\frac{1}{2}\right] = \frac{1}{2} \Rightarrow \boxed{k = 1}$$

(ii) $P[0.2 < X < 1.2] = \int_{0.2}^{1.2} f(x)\,dx$

$$= \int_{0.2}^{1} x\,dx + \int_{1}^{1.2} (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_{0.2}^{1} + \left[2x - \frac{x^2}{2}\right]_1^{1.2}$$

$$= \frac{1}{2}\left[1 - 0.04\right] + \left[\left(2.4 - 0.72\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[0.96\right] + \left[0.18\right] = 0.66$$

(iii) $P[0.5 < X < 1.5 / X \geq 1]$

$$= \frac{P[1 < X < 1.5]}{P[X \geq 1]} \rightarrow (1)$$

$$P[1 < X < 1.5] = \int_1^{1.5} (2-x)\,dx = \left[2x - \frac{x^2}{2}\right]_1^{1.5}$$

$$= \left[\left(3 - \frac{2.25}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \left[3 - 1.125 - 1.5\right] = 0.375$$

$$P[X \geq 1] = \int_1^2 (2-x)\,dx = \left[2x - \frac{x^2}{2}\right]_1^2 = \left[(4-2) - \left(2 - \frac{1}{2}\right)\right] = 0.5$$

sub. in (1)

$$P[0.5 < X < 1.5 / X \geq 1] = \frac{0.375}{0.5} = 0.75$$

(iv) To find CDF

<u>when $0 < x < 1$</u>

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{x} x\,dx = \left(\frac{x^2}{2}\right)_0^x$$

$$= \frac{x^2}{2}$$

<u>when $1 < x < 2$</u>

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{1} x\,dx + \int_{1}^{x} (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left[2x - \frac{x^2}{2}\right]_1^x$$

$$= \frac{1}{2} + \left[\left(2x - \frac{x^2}{2}\right) - \left(2 - \frac{1}{2}\right)\right]$$

$$= \frac{1}{2} - \frac{3}{2} + 2x - \frac{x^2}{2}$$

$$= 2x - \frac{x^2}{2} - 1$$

<u>when $x > 2$</u>

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx + \int_{2}^{7} 0\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + \left(2x - \frac{x^2}{2}\right)_1^2 = \frac{1}{2} + \left(4 - \frac{1}{2}2\right) - \left(2 - \frac{1}{2}\right)$$

$$= \frac{1}{2} + 2 - \frac{3}{2} = 1$$

$$\therefore F(x) = \begin{cases} 0 & , x < 0 \\ x^2/2 & , 0 < x < 1 \\ 2x - \frac{x^2}{2} - 1 & , 1 < x < 2 \\ 1 & , x > 2 \end{cases}$$

(11) If the density function of a continuous r.v X is given by $f(x) = \begin{cases} ax & , 0 \le x \le 1 \\ a & , 1 \le x \le 2 \\ 3a - ax & , 2 \le x \le 3 \\ 0 & , \text{otherwise} \end{cases}$

Find (i) the value of 'a'

(ii) cdf of X.

(i) WKT $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a - ax)\,dx = 1$

$$a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + \left[3ax - \frac{ax^2}{2}\right]_2^3 = 1$$

$$\Rightarrow a\left(\frac{1}{2}\right) + a + \left[\left(9a - \frac{9a}{2}\right) - (6a - 2a)\right] = 1$$

$$\Rightarrow \frac{3a}{2} + \left[5a - \frac{9a}{2}\right] = 1 \Rightarrow \frac{3a}{2} + \frac{a}{2} = 1$$

$$\Rightarrow 2a = 1 \Rightarrow \boxed{a = 1/2}$$

(ii) <u>To find CDF</u>

<u>when $0 \le x \le 1$</u> $F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^x \frac{x}{2}\,dx$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^x = \frac{x^2}{4}.$$

when $1 \le x \le 2$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^1 \frac{x}{2}\,dx + \int_1^x \frac{1}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}\,(x)_1^x$$

$$= \frac{1}{4} + \frac{1}{2}(x-1) = \frac{1}{2}\left(x - \frac{1}{2}\right)$$

when $2 \le x \le 3$

$$F(x) = \int_{-\infty}^{x} f(x)\,dx = \int_0^1 \frac{x}{2}\,dx + \int_1^2 \frac{1}{2}\,dx + \int_2^x \left(\frac{3}{2} - \frac{x}{2}\right)dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2}\right)_0^1 + \frac{1}{2}(x)_1^2 + \left[\frac{3}{2}x - \frac{x^2}{4}\right]_2^x$$

$$= \frac{1}{4} + \frac{1}{2} + \left[\left(\frac{3}{2}x - \frac{x^2}{4}\right) - (3-1)\right]$$

$$= \frac{1}{4} + \frac{1}{2} - 2 + \frac{3}{2}x - \frac{x^2}{4}$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$$

$$F(x) = \begin{cases} 0, & x < 0 \\ \frac{x^2}{4}, & 0 \le x \le 1 \\ \frac{1}{2}\left(x - \frac{1}{2}\right), & 1 \le x \le 2 \\ \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}, & 2 \le x \le 3 \\ 1, & x > 3 \end{cases}$$