TOPIC : 1.7 – Binomial & Poisson Distribution

Standard Distribution (Discrete)

1. Binomial distribution
2. Poisson distribution
3. Geometric distribution

Binomial Distribution

Binomial distribution is derived from experiment known as Bernoulli trial.

A random experiment whose outcomes can be classified into two categories, usually called 'success' and 'failure', is called a Bernoulli trail.

A random variable X is said to follow binomial distribution with parameter n and p if its pmf is given by

$$P[X=x] = nC_x \, p^x q^{n-x}, \quad x = 0, 1, 2, \dots$$

X is called a binomial random variable.

where n - no. of trials
p - probability of success.
q - probability of failure

$p+q = 1$

1. Find the MGF of the binomial distribution and hence find mean and variance.

$$P[X=x] = nc_x \, p^x q^{n-x}$$

$$MGF = M_X(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{n} e^{tx} \, nc_x \, p^x q^{n-x}$$

$$= q^n + nc_1 \, pe^t q^{n-1} + nc_2 \, (pe^t)^2 q^{n-2} + \cdots + nc_n (pe^t)^n$$

$$= (pe^t + q)^n$$

$$(x+y)^n = x^n + nc_1 x^{n-1} y + nc_2 x^{n-2} y^2 \cdots + nc_n y^n$$

$$\text{Mean} = \left\{ \frac{d}{dt} [M_X(t)] \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} (pe^t + q)^n \right\}_{t=0}$$

$$= \left\{ n (pe^t + q)^{n-1} (pe^t) \right\}_{t=0}$$

$$= n (p+q)^{n-1} p$$

$$nc_r = 1c_1$$

$$= np$$

$$E[X^2] = \left\{ \frac{d^2}{dt^2} [M_X(t)] \right\}_{t=0}$$

$$= \left\{ \frac{d}{dt} np \, e^t (pe^t + q)^{n-1} \right\}_{t=0}$$

$$= np\left[e^{t}(n-1)(pe^{t}+q)^{n-2}(pe^{t}) + (pe^{t}+q)\right]$$

$$= np\left[(n-1)(p+q)^{n-2}p + (p+q)\right]$$

$$= np\left[(n-1)p+1\right] = np\left[np-p+1\right]$$

$$= np\left[np+q\right] = n^2p^2 + npq$$

Variance $= E[X^2] - [E(X)]^2$

$$= n^2p^2 + npq - n^2p^2$$

$$= npq$$

2. For a Binomial distribution with mean 6 & standard deviation $\sqrt{2}$, find the first two terms

Gn. Mean $= np = 6 \rightarrow (1)$

Variance $= npq = 2 \rightarrow (2)$

(2)/(1) $\Rightarrow \dfrac{npq}{np} = \dfrac{2}{6} = \dfrac{1}{3}$

$\Rightarrow \boxed{q = \frac{1}{3}}$ $\qquad p = 1-q$

$\boxed{p = \frac{2}{3}}$

$np = 6$

$n\left(\frac{2}{3}\right) = 6$

$\boxed{n = 9}$

$P(X=x) = nC_x\, p^x q^{n-x}, \quad x = 0$

$$P[X=0] = 9C_0\left(\frac{2}{3}\right)^0\left(\frac{1}{3}\right)^9$$

$$= \left(\frac{1}{3}\right)^9$$

$$P[X=1] = 9C_1\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^8$$

$$= 18\left(\frac{1}{3}\right)^9$$

(8) In a large consignment of electric bulb 10% are defective. A random sample of 20 is taken for inspection. Find the prob. that
(i) All are good bulbs (ii) Atmost there are 3 defective bulbs (iii) Exactly there are 3 defective bu

Let x denote the no. of defective bulbs.
p - the prob. that an electric bulb is defective $= \frac{1}{10}$

$q = 1 - p = \frac{9}{10}$ and $n = 20$.

Binomial distribution $P[x = x] = nc_x \, p^x q^{n-x}$

$x = 0, 1, 2, \cdots$

(i) All are good bulbs:

$= P[x = 0] = 20c_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{20} = \left(\frac{9}{(10)}\right)^{20}$

$= 0.1216$

(ii) Almost there are 3 defective bulbs.

$P[x \le 3] = P[x = 0] + P[x = 1] + P[x = 2] + P[x =$

$= 20c_0 \left(\frac{1}{10}\right)^0 \left(\frac{9}{10}\right)^{20} + 20c_1 \left(\frac{1}{10}\right)\left(\frac{9}{10}\right)^{19}$

$+ 20c_2 \left(\frac{1}{10}\right)^2 \left(\frac{9}{10}\right)^{18} + 20c_3 \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{17}$

$= \left(\frac{9}{10}\right)^{20} + \cancel{20} \frac{9^{19}}{10^{20}} \; 2 \left(\frac{9}{10}\right)^{19} + \frac{190}{100} \left(\frac{9}{10}\right)^{18} +$

$= 0.2666$

(iii) Exactly 3 are defective

$P[x = 3] = 20c_3 \left(\frac{1}{10}\right)^3 \left(\frac{9}{10}\right)^{17}$

$= 0.19$

1) Out of 800 families with 4 children each, how many families would be expected to have
(i) 2 boys and 2 girls ; (ii) atleast 1 boy
(iii) atmost 2 girls (iv) children of both genders.
Assume equal probabilities for boys and girls.

Considering each child is a trial, $n = 4$.
Assuming that birth of a boy is a success,
$p = \frac{1}{2}$ and $q = \frac{1}{2}$.

Let x denote the no. of success (boys).
Binomial distribution $P[X = x] = nc_x\, p^x q^{n-x}$
$x = 0, 1, 2, 3, 4$

i) P [at 2 boys and 2 girls]

$= P[X = 2] = 4c_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 6 \left(\frac{1}{2}\right)^4$

$= \frac{6}{16} = \frac{3}{8}$

∴ No. of families having 2 boys & 2 girls

$= 800 \times \frac{3}{8} = 300$

(ii) $P[\text{atleast 1 boy}] = P[X \geq 1] = 1 - P[X < 1]$

$= 1 - P[X = 0] = 1 - 4c_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4$

$= 1 - \frac{1}{2^4} = 1 - \frac{1}{16} = \frac{15}{16}$

∴ No. of families having atleast 1 boy

$= 800 \times \frac{15}{16} = 750$

(iii) $P[\text{atmost } 2 \text{ girls}] = P[\text{atleast } 2 \text{ boys}]$

$$= P[x \geq 2] = 1 - P[x < 2]$$

$$= 1 - \left\{ P[x=0] + P[x=1] \right\}$$

$$= 1 - \left\{ 4C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4 + 4C_1 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)^3 \right\}$$

$$= 1 - \left\{ \left(\tfrac{1}{2}\right)^4 + 4\left(\tfrac{1}{2}\right)^4 \right\}$$

$$= 1 - \frac{1}{16}(5) = \frac{11}{16}$$

$\therefore$ No. of families having atmost 2 girls

$$= 800 \times \frac{11}{16} = 550$$

(iv) $P[\text{children of both genders}]$

$$= 1 - P[\text{children of the same gender}]$$

$$= 1 - \left\{ P[\text{all are boys}] + P[\text{all are girls}] \right\}$$

$$= 1 - \left\{ P[x=4] + P[x=0] \right\}$$

$$= 1 - \left\{ 4C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^0 + 4C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^4 \right\}$$

$$= 1 - \left\{ \left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^4 \right\}$$

$$= 1 - \frac{2}{2^4} = 1 - \frac{1}{2^3} = 1 - \frac{1}{8} = \frac{7}{8}$$

$\therefore$ No. of families having children of both genders

$$= 800 \times \frac{7}{8} = 700$$

Poisson Distribution.

A random variable X is said to follow Poisson distribution if it assumes only non-negative values and its pmf is given by

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\dots$$
$$\lambda > 0$$

λ is known as the parameter of the Poisson distribution. $(\lambda = np)$.

1) Find the MGF of the Poisson distribution and hence find mean and variance.

The pmf of the Poisson distribution

$$P[X=x] = \frac{e^{-\lambda}\lambda^{x}}{x!}, \quad x = 0,1,2,\dots$$

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MGF $M_X(t) = E[e^{tX}] = E$

$$= \sum_{x=0}^{\infty} e^{tx} p(x)$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^{t})^{x}}{x!}$$

$$= e^{-\lambda}\left[1 + \frac{(\lambda e^{t})}{1!} + \frac{(\lambda e^{t})^{2}}{2!} + \dots\right]$$

$$= e^{-\lambda} e^{\lambda e^{t}} = e^{\lambda(e^{t}-1)}$$

$$\text{Mean} = E[x] = \left\{\frac{d}{dt}\left(M_x(t)\right)\right\}_{t=0}$$

$$= \left[\frac{d}{dt} e^{\lambda(e^t-1)}\right]_{t=0} = \left\{e^{\lambda(e^t-1)} \lambda e^t\right\}_{t=0}$$

$$\boxed{\text{Mean} = \lambda}$$

$$E[x^2] = \left\{\frac{d^2}{dt^2} M_x(t)\right\}_{t=0}$$

$$= \lambda \left\{\frac{d}{dt} e^t e^{\lambda(e^t-1)}\right\}_{t=0}$$

$$= \lambda \left[e^t e^{\lambda(e^t-1)} \lambda e^t + e^{\lambda(e^t-1)} e^t\right]_{t=0}$$

$$= \lambda[\lambda+1] = \lambda^2 + \lambda$$

$$\text{Var}(x) = E[x^2] - [E(x)]^2$$

$$= \lambda^2 + \lambda - \lambda^2$$

$$\boxed{\text{Var}(x) = \lambda}$$

1. Every week the average no. of wrong-number phone calls received by a certain mail order house seven. What is the Prob. that they will receive 2 wrong calls tomorrow?

The average no. of wrong-number phone calls received in a week $= 7$

Average number of wrong-number calls per day $= \frac{7}{7} = 1 = \lambda$.

Let x denote the no. of wrong-number phone calls per day. The pmf of Poisson $P[x=x] = \frac{e^{-\lambda}\lambda^x}{x!}$

$$\therefore P[x=2] = \frac{e^{-1}(1)^2}{2!} = \frac{e^{-1}}{2} =$$

④ The number of monthly breakdown of a computer is a random variable having a Poiss dis. with mean = 1.8. Find the prob. that this computer will function for a month
(i) without a breakdown (ii) with only one break and (iii) with atleast one breakdown.

Given Mean $= \lambda = 1.8$

Let x denote the no. of breakdowns of a computer in a month.

The pmf of Poisson dis. $P[x=x] = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$

$x = 0, 1, 2, \ldots$

(i) P[with a breakdown]

$= P[x=0] = \dfrac{e^{-1.8}(1.8)^{0}}{0!} = e^{-1.8} = 0.1653$

P[with only one breakdown]

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$= \dfrac{e^{-1.8}(1.8)^{1}}{1!} = 0.2975$

) P[with atleast 1 breakdown] $= P[x \geq 1]$

$= 1 - P[x<1] = 1 - P[x=0] = 1 - 0.1653$

$= 0.8347$