TOPIC : 1.9 – Geometric & Uniform Distribution

Geometric Distribution

Suppose that If the random Variable X denotes the number of trials required for the first success in independent repeated Bernoulli trials with probability of success p, then

$$P[X=x] = q^{x-1} p \quad , \quad x = 1, 2, 3, \ldots$$

1. Find the MGF and mean, variance of the Geometric distribution.

The pmf of the Geometric distribution

$$P[X=x] = q^{x-1} p \quad , \quad x = 1, 2, 3, \ldots$$

MGF $\quad M_X(t) = E[e^{tx}] = \sum_{x=1}^{\infty} e^{tx} p(x)$

$$= \sum_{x=1}^{\infty} e^{tx} q^{x-1} p$$

$$= \sum_{x=1}^{\infty} e^{t} \cdot e^{t(x-1)} q^{x-1} p$$

$$= pe^t \sum_{x=1}^{\infty} (qe^t)^{x-1}$$

$$= pe^t \left[1 + qe^t + (qe^t)^2 + \cdots\right]$$

$$= pe^t (1 - qe^t)^{-1}$$

$$M_X(t) = \frac{pe^t}{1 - qe^t}$$

$$\text{Mean} = \left[\frac{d}{dt}\{M_X(t)\}\right]_{t=0} = \left[\frac{d}{dt}\left\{\frac{pe^t}{1-qe^t}\right\}\right]_{t=0}$$

$$= \left[\frac{(1-qe^t)pe^t - pe^t(-qe^t)}{(1-qe^t)^2}\right]_{t=0}$$

$$= \left[\frac{(1-q)p + pq}{(1-q)^2}\right] = \left[\frac{p - pq + pq}{p^2}\right] = \frac{p}{p^2}$$

$$\boxed{\text{Mean} = \frac{1}{p}}$$

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$$\text{Var } E[x^2] = \left[\frac{d^2}{dt^2}\left\{M_x(t)\right\}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\frac{(1-qe^t)\,pe^t + pqe^{2t}}{(1-qe^t)^2}\right\}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\frac{pe^t - pqe^{2t} + pqe^{2t}}{(1-qe^t)^2}\right\}\right]_{t=0}$$

$$= \left[\frac{d}{dt}\left\{\frac{pe^t}{(1-qe^t)^2}\right\}\right]_{t=0}$$

$$= p\left[\frac{(1-qe^t)^2 e^t - e^t\, 2(1-qe^t)(-qe^t)}{(1-qe^t)^4}\right]_{t=0}$$

$$= p\left[\frac{(1-q)^2 + 2(1-q)q}{(1-q)^4}\right]$$

$$= p\left[\frac{p^2+2pq}{p^4}\right] = \frac{p^2+2pq}{p^3}$$

$$= \cancel{\frac{p(p+2q)}{p^3}} = \cancel{\frac{p+2q}{p^2}} = \cancel{p+2(1/p)}$$

$$\cancel{p^2+2p(1-q)} = \frac{p(p+2q)}{p^3}$$

$$= \frac{[1-q+2q]}{p^2} = \frac{1+q}{p^2} = \frac{1}{p^2} + \frac{q}{p^2}$$

$$\text{Var}(x) = E[x^2] - [E(x)]^2 = \frac{1}{p^2} + \frac{q}{p^2} - \frac{1}{p^2}$$

$$\boxed{\text{Var}(x) = \frac{q}{p^2}}$$

3. Suppose that a trainee soldier shoots a target in an independent fashion. If the prob. that the target is shot on any shot is 0.8, what is the prob. that (i) the target would be hit on 6th attempt
(ii) it takes him less than 5 shots
(iii) it takes him an even no. of shots.

Given $p = 0.8$

$q = 1 - p = 0.2$

The pmf of the Geometric dis.

$$P[X = x] = q^{x-1} p, \quad x = 1, 2, 3, \ldots$$

(i) $P[\text{hit on the } 6^{th} \text{ attempt}] = P[X = 6]$

$= (0.2)^5 (0.8) = 0.000256$

(ii) $P[\text{less than 5 shots}] = P[X < 5]$

$= P[X=1] + P[X=2] + P[X=3] + P[X=4]$

$= (0.2)^0(0.8) + (0.2)(0.8) + (0.2)^2(0.8) + (0.2)^3(0.8)$

$= (0.8)[1 + 0.2 + 0.04 + 0.006]$

$= (0.8)[1.246] = 0.9984$

(iii) $P[\text{even no. of shots}] = P[X=2] + P[X=4] + P[X=6] + \cdots$

$= (0.2)(0.8) + (0.2)^3(0.8) + (0.2)^5(0.8) + \cdots$

$= (0.2)(0.8)[1 + (0.2)^2 + (0.2)^4 + \cdots]$

$= (0.16)[1 + (0.04) + (0.04)^2 + \cdots]$

$= (0.16)[1 - 0.04]^{-1} = \dfrac{0.16}{0.96} = 0.1667$

∴ If the prob. that an applicant for a driving licence will pass the road test on any given trial is 0.8, what is the prob. that he will finally pass the test (i) on the fourth trial & (ii) in fewer than 4 trials?

Let x denote the no. of trials reqd. to achieve the first success.

The pmf of the geometric dis. 30

$$P[X = x] = q^{x-1} p \quad . \quad x = 1, 2, \ldots$$

Here $p = 0.8$; $q = 1 - p = 0.2$

(i) $P[X = 4] = (0.2)^3 (0.8) = (0.008)(0.8)$

$= 0.0064$

(ii) $P[X < 4] = P[X = 1] + P[X = 2] + P[X = 3]$

$= (0.2)^0 (0.8) + (0.2)(0.8) + (0.2)^2 (0.8)$

$= (0.8) [1 + 0.2 + 0.04]$

$= (0.8)(1.24) = 0.9984$

Standard Distributions (continuous case)

1. Uniform distribution
2. Exponential distribution
3. Gamma distribution
4. Normal distribution

Uniform Distribution 32

A continuous random variable X is said to follow a uniform distribution over an interval (a,b) if its pdf is given by

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

'a' and 'b' are called the parameters of x.

① Find the MGF of an uniform distribution and hence find mean and variance.

The pdf of a uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$$

MGF $M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$$= \int_a^b e^{tx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{(b-a)t}\left[e^{bt} - e^{at}\right]$$

$$\boxed{M_x(t) = \frac{e^{bt} - e^{at}}{(b-a)t}}$$

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(ii) $P[x \geq 3] = 1 - P[x < 3]$

$= 1 - \{P[x=0] + P[x=1] + P[x=2]\}$

$= 1 - \{6C_0 (\frac{2}{3})^0 (\frac{1}{3})^6 + 6C_1 (\frac{2}{3})(\frac{1}{3})^5 + 6C_2 (\frac{2}{3})^2$

$= 1 - \{(\frac{1}{3})^6 + \frac{12}{3^6} + 15 \cdot \frac{4}{3^6}\}$

$= 1 - \frac{1}{3^6}[1 + 12 + 60] = 1 - \frac{73}{3^6}$

$= 0.8998$

(iii) $P[x \leq 4] = 1 - P[x \geq 4]$

$= 1 - \{P[x=5] + P[x=6]\}$

$= 1 - \{6C_5 (\frac{2}{3})^5 (\frac{1}{3}) + 6C_6 (\frac{2}{3})^6 ($

$= 1 - \{6 \frac{2^5}{3^6} + \frac{2^6}{3^6}\}$

$= 1 - \frac{1}{3^6}\{192 + 64\}$

$= 1 - \frac{256}{3^6} = 0.6488$

Buses arrive at a specified bus stop at ulis intervals starting at 7 a.m., 7.15 a.m, 7.30 a.m. a passenger arrives at the bus stop at a r which is uniformly distributed between 7 and the prob. that he waits (i) less than 5 mini atleast 12 min for a bus.

et x denotes the time that a passenger arri ween 7 and 7.30 am.

Then x follows uniform dis. over (0,3

pdf of x $f(x) = \frac{1}{b-a} = \frac{1}{30}, \quad 0 < x < 30$

Passenger waits less than 5 minutes, he arrives between 7.10 – 7.15 or 25 – 7.30.

waiting time less than 5 min]

$= P[10 \leq x \leq 15] + P[25 \leq x \leq 30]$

$= \int_{10}^{15} f(x)\,dx + \int_{25}^{30} f(x)\,dx = \frac{1}{30}(x)_{10}^{15} + \frac{1}{30}(x)$

$= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

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(ii) Passenger waits atleast 12 min i) he arrives between 7 - 7.03 (or) 7.15 - 7.18

P [waiting time atleast 12 min.]

$= P[0 \le X \le 3] + P[15 \le X \le 18]$

$= \int_0^3 \frac{1}{30}\,dx + \int_{15}^{18} \frac{1}{30}\,dx = \frac{1}{30}(x)_0^3 + \frac{1}{30}(x)_{15}^{18}$

$= \frac{1}{10} + \frac{1}{10} = \frac{1}{5}$