# SNS COLLEGE OF ENGINEERING

**(Autonomous)**

**DEPARTMENT OF ELECTRONICS AND COMMUNICATION ENGINEERING**

# UNIT- I

# Discrete Fourier Transform

## Properties of DFT

# Properties of DFT

- Linearity
- Periodicity
- Circular Time Shift
- Time Reversal
- Conjugation
- Circular frequency Shift
- Multiplication
- Circular Convolution
- Circular Correlation
- Parseval's Theorem

# Linearity

**Statement:** The linearity property of DFT states that the DFT of a linear weighted combination of two or more signals is equal to similar linear weighted combination of the DFT of individual signals.

Let $DFT\{x_1(n)\}=X_1(K)$ & $DFT\{x_2(n)\}=X_2(K)$ then,

$$DFT\{a_1x_1(n)+a_2x_2(n)\}=a_1X_1(K)+a_2X_2(K)$$

Where $a_1$ & $a_2$ are constants.

**Proof:** By definition of DFT,

$X_1(K)=DFT\{x_1(n)\}=\sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}$-------(1)

$X_2(K)=DFT\{x_2(n)\}=\sum_{n=0}^{N-1} x_2(n)\, e^{-j2\pi kn/N}$-------(2)

$$DFT\{a_1x_1(n)+a_2x_2(n)\}= \sum_{n=0}^{N-1} \{a_1x_1(n)+a_2x_2(n)\}\, e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} \{a_1x_1(n)\, e^{-j2\pi kn/N}+a_2x_2(n)\, e^{-j2\pi kn/N}\}$$

$$=a_1 \sum_{n=0}^{N-1} \{x_1(n)\}\, e^{-j2\pi kn/N}+a_2 \sum_{n=0}^{N-1}\{ x_2(n)\}\, e^{-j2\pi kn/N}$$

$DFT\{a_1x_1(n)+a_2x_2(n)\}= a_1X_1(K)+a_2X_2(K)$ [By Equ 1 and 2]

# Periodicity

**Statement:** If a sequence x(n) is periodic with periodicity of N samples then N-point DFT, X(K) is also periodic with a periodicity of N Samples

Let $x(n)=x(n+N)$ for all n

$x(K)=x(K+N)$ for all K

**Proof:** By definition of DFT, the(K+N)$^{th}$ coefficient of X(K) is given by,

$$X(K+N)=\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi(k+N)n/N}$$

$$=\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}.\, e^{-j2\pi Nn/N}$$

$$=\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}.1 \qquad [e^{-j2\pi n} = 1]$$

$$X(K+N)=X(K)$$

# Circular Time Shift

**Statement:** The Circular time shift property of DFT says that if a discrete time signal is circularly shifted in the time by 'm' units then its DFT is multiplied by $e^{-j2\pi km/N}$

Let DFT{x(n)}=X(K) then,

$X(K).\, e^{-j2\pi km/N} = DFT\{x(n-m)_N\}$

**Proof:**

DFT$\{x(n-m)_N\}=\sum_{n=0}^{N-1} x\{((n-m))_N\}\, e^{-j2\pi kn/N}$

Let p=n-m

n=m+p

DFT$\{x(n-m)_N\}$ $=\sum_{p=0}^{N-1} x(p)\, e^{-j2\pi k(m+p)/N}$

$=\sum_{p=0}^{N-1} x(p)\, e^{-j2\pi km/N} . e^{-j2\pi kp/N}$

$=e^{-j2\pi km/N} \sum_{p=0}^{N-1} x(p) . e^{-j2\pi kp/N}$

DFT$\{x(n-m)N\} = e^{-j2\pi km/N}$ . X(K)

# Conjugation

**Statement:** Let $x(n)$ be a complex N-point discrete sequence $x^*(n)$ be its conjugate sequence

If $\quad DFT\{x(n)\}=X(K)$ then,

$$DFT\{x^*(n)\}=X^*(N-K)$$

**Proof:**

$$\text{DFT}\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\ e^{-j2\pi kn/N}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} \cdot 1\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} \cdot e^{-j2\pi n}\right]^* \{e^{-j2\pi n} = 1\}$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{j2\pi kn/N} \cdot e^{-j2\pi nN/N}\right]^*$$

$$= \left[\sum_{n=0}^{N-1} x(n)\ e^{-j2\pi(N-k)n/N}\right]^*$$

$$\text{DFT}\{x^*(n)\} = X^*(N-k)$$

# Multliplication

**Statement:** The Multiplication Property of DFT says that the DFT of product of two discrete time sequences is equivalent to circular convolution of DFT's of the individual sequences scaled by the factor of 1/N

If DFT{x(n)}=X(K) then,

$DFT\{x_1(n)\, x_2(n)\}=1/N\{X_1(K)\Theta X_2(K)\}$

**Proof:**

By definition of IDFT,

$x_1(n)=\frac{1}{N}\sum_{k=0}^{N-1} X_1(K)\, e^{j2\pi kn/N}$

Let k=m

$x_1(n)=\frac{1}{N}\sum_{m=0}^{N-1} X_1(m)\, e^{j2\pi mn/N}$ --------------(1)

By definition of DFT,

$DFT\{x_1(n)\, x_2(n)\} = \sum_{n=0}^{N-1} x_1(n)\, x_2(n) e^{-j2\pi kn/N}$ -----------(2)

Substitute (1) in (2) we get,

$DFT\{x_1(n)\, x_2(n)\} = \{ \sum_{n=0}^{N-1} [1/N \sum_{m=0}^{N-1} X_1(m)\, e^{j2\pi mn/N}]\, x_2(n) e^{-j2\pi kn/N}\}$

$$DFT\{x_1(n)x_2(n)\}=\{1/N\sum_{m=0}^{N-1} X_1(m)[\sum_{n=0}^{N-1} x_2(n)\ e^{-j2\pi kn/N}\ e^{j2\pi mn/N}\}$$

$$=\{1/N\sum_{m=0}^{N-1} X_1(m)[\sum_{n=0}^{N-1} x_2(n)\ e^{-j2\pi(k-m)n/N}\}$$

$$DFT\{x_1(n)x_2(n)\}=\{1/N\sum_{m=0}^{N-1} X_{1(m)}\, X_2((k-m))_N\}$$

$$= 1/N\{X_1(K)\Theta X_2(K)\}$$

# Circular Convolution

**Statement:** The Circular Convolution of two N-Point Sequences $x_1(n)$ & $x_2(n)$ is defined as,

$x_1(n) \Theta x_2(n) = \sum_{n=0}^{N-1} x_1(n)\, x_2((n-m))_N$

The Convolution Property of DFT says that, the DFT of circular convolution of two sequences is equivalent to product of their individual DFTs

Let DFT{$x_1(n)$}=$X_1(K)$ and DFT{$x_2(n)$}=$X_2(K)$ then,

By Convolution property,

DFT{$x_1(n) \Theta x_2(n)$}=$X_1(K)\, X_2(K)$

Proof: Let $x_1(n)$ & $x_2(n)$ be N-Point Sequences, Now by definition of DFT

$X_1(K) = \sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N} = \sum_{m=0}^{N-1} x_1(m)\, e^{-j2\pi km/N}$ [n=m] – (1)

$X_2(K) = \sum_{n=0}^{N-1} x_2(n)\, e^{-j2\pi kn/N} = \sum_{p=0}^{N-1} x_2(p)\, e^{-j2\pi kp/N}$ [n=p]---(2)

Consider the Product of $X_1(K)\, X_2(K)$. The inverse DFT of the product is given by,

$DFT^{-1}\{X_1(K)\, X_2(K)\} = 1/N \sum_{k=0}^{N-1} X_1(K) X_2(K)\, e^{j2\pi kn/N}$-------(3)

Substitute the value of equ 1 and 2 in equ 3 We get,

$DFT^{-1}\{X_1(K)$

$X_2(K)\} = \frac{1}{N} \sum_{n=0}^{N-1} [\sum_{m=0}^{N-1} x_1(m)\, e^{-j2\pi km/N}]\, [\sum_{p=0}^{N-1} x_1(m)\, e^{-j2\pi kp/N}] e^{j2\pi kn/N}$

$\text{DFT}^{-1}\{X_1(K)\,X_2(K)\} = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m) \sum_{p=0}^{N-1} x_2(p) \sum_{k=0}^{N-1} e^{j2\pi k(n-m-p)/}N$---(4)

Consider the Summation $\sum_{k=0}^{N-1} e^{j2\pi k(n-m-p)/}N$ in equ (4)

Let n-m-p=qN and q is an integer

$\sum_{k=0}^{N-1} e^{j2\pi k(n-m-p)/}N = \sum_{k=0}^{N-1} e^{j2\pi kqN/}N = \sum_{k=0}^{N-1} e^{j2\pi kq} = \sum_{k=0}^{N-1} e^{(j2\pi q)^K}$

$= \sum_{k=0}^{N-1} (1)^K = N$------(5)

Consider the summation $\sum_{p=0}^{N-1} x_2(p)$ in the equ (4)we get

$\sum_{p=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2(n-m-qN) = \sum_{m=0}^{N-1} x_2(n-m, Mod\ N)$

$\sum_{m=0}^{N-1} x_2(p) = \sum_{m=0}^{N-1} x_2((n-m))_N$-------(6)

$$DFT^{-1}\{X_1(K)\, X_2(K)\} = 1/N \sum_{m=0}^{N-1} x_1(m) \sum_{m=0}^{N-1} x_2((n-m))_N \,.N$$

$$DFT^{-1}\{X_1(K)\, X_2(K)\} = \sum_{m=0}^{N-1} x_1(m)\; x_2((n-m))_N$$

$$DFT^{-1}\{X_1(K)\, X_2(K)\} = x_1(n) \,\Theta\, x_2(n)$$

$$X_1(K)\, X_2(K) = DFT\{x_1(n) \,\Theta\, x_2(n)\}$$

Thus proved

# Circular Correlation

**Statement:** The Circular Correlation of two sequence x(n) and y(n) is defined as

$r_{xy}(m) = \sum_{n=0}^{N-1} x(n)\, y^*((n-m))_N$

Let DFT{x(n)}=X(K) and DFT{y(n)}=Y(K) then by Correlation property,

$X(K)=\sum_{n=0}^{N-1} x(n)\, e^{-j2\pi kn/N}$-----(1)

$Y(K)=\sum_{p=0}^{N-1} y(p)\, e^{-j2\pi kp/N}$ [n=p]---(2)

Consider the product of $X(K)\, Y^*(K)$. The IDFT of the product is given by,

$DFT^{-1}\{X(K)\, Y^*(K)\}=1/N \sum_{K=0}^{N-1} X(K)\, Y^*(K)\, e^{j2\pi kn/N}$

Let n=m

$$=1/N \sum_{K=0}^{N-1} X(K)\, Y^*(K)\, e^{j2\pi km/N} \text{--------(3)}$$

Substitute the value of equ 1 and 2 in equ 3 We get,

$$=\frac{1}{N}\sum_{K=0}^{N-1}\left[\sum_{n=0}^{N-1}x(n)\,e^{-j2\pi kn/N}\right]\left[\sum_{p=0}^{N-1}y(p)\,e^{-j2\pi kp/N}\right]e^{j2\pi km/N}$$

$$=\frac{1}{N}\sum_{n=0}^{N-1}x(n)\sum_{p=0}^{N-1}y^{*}(p)\sum_{k=0}^{N-1}e^{j2\pi k(m-n+p)/N} \text{ ------(4)}$$

Consider the Summation $\sum_{k=0}^{N-1}e^{j2\pi k(n-m+p)/N}$ in equ (4)

Let n-m+p=qN and q is an integer

$$\sum_{k=0}^{N-1}e^{j2\pi k(n-m+p)/N}=\sum_{k=0}^{N-1}e^{j2\pi kqN/N}=\sum_{k=0}^{N-1}e^{j2\pi kq}=\sum_{k=0}^{N-1}e^{(j2\pi q)^{K}}$$

$$=\sum_{k=0}^{N-1}(1)^{K}=N\text{------(5)}$$

Consider the summation $\sum_{p=0}^{N-1}y^{*}(p)$ in the equ (4)we get

$$\sum_{p=0}^{N-1}y^{*}(p)=\sum_{n=0}^{N-1}y^{*}(n-m+qN)=\sum_{n=0}^{N-1}y^{*}(n-m,Mod\ N)$$

$$\sum_{p=0}^{N-1}y^{*}(p)=\sum_{n=0}^{N-1}y^{*}((n-m))_{N}\text{-------(6)}$$

$DFT^{-1}\{X(K)\ Y^*(K)\}=1/N \sum_{n=0}^{N-1} x(n) \sum_{n=0}^{N-1} y^*((n-m))_N .N$

$DFT^{-1}\{X(K)\ Y^*(K)\}=\sum_{n=0}^{N-1} x(n) \sum_{n=0}^{N-1} y^*((n-m))_N$

$=\sum_{n=0}^{N-1} x(n)\ y^*((n-m))_N$

$DFT^{-1}\{X(K)\ Y^*(K)\}= r_{xy}(m)$

$X(K)\ Y^*(K)=DFT\{r_{xy}(m)\}$

**Thus proved**

# Parseval's Theorem

**Statement:** Let DFT{$x_1(n)$}=$X_1(K)$& DFT{$x_2(n)$}=$X_2(K)$ then by Parseval' theorem

$$\sum_{n=0}^{N-1} x_{1(n)}\, x_2^{\,*}(n) = 1/N \sum_{K=0}^{N-1} X_1(K)\, X_2^*(K)$$

**Proof:**

Let $x_1(n)$ and $x_2(n)$ be N-point Sequences

By the definition of DFT, $X_1(K)=\sum_{n=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}$----(1)

By the definition of IDFT, $x_{2(n)} = 1/N \sum_{k=0}^{N-1} X_2(K)\, e^{j2\pi kn/N}$---(2)

Consider the right hand side term of the Parseval's theorem,

$$1/N \sum_{K=0}^{N-1} X_1(K)\, X_2^*(K) = \frac{1}{N} \sum_{n=0}^{N-1} [\sum_{K=0}^{N-1} x_1(n)\, e^{-j2\pi kn/N}] X_2^*(K)$$

$$1/N \sum_{K=0}^{N-1} X_1(K)\, X_2^*(K) = \sum_{n=0}^{N-1} x_1(n) [\frac{1}{N} \sum_{K=0}^{N-1} X_2^*(K) e^{-j2\pi kn/N}]$$

$$1/N \sum_{K=0}^{N-1} X_1(K)\, X_2^*(K) = \sum_{n=0}^{N-1} x_1(n) [\frac{1}{N} \sum_{K=0}^{N-1} X_2(K) e^{j2\pi kn/N}]^*$$

$$1/N \sum_{K=0}^{N-1} X_1(K)\, X_2^*(K) = \sum_{n=0}^{N-1} x_{1(n)}\, x_2^*(n)$$

Thus Proved.

# Thank You!