TOPIC : 2.6.INTERPOLATION WITH EQUAL INTERVALS - CUBIC SPLINES

INTERPOLATING WITH A CUBIC SPLINE:

We know that

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right], \text{ for } i = 1, 2, 3, \dots, n-$$

Here $M_0 = 0$ and $M_n = 0$

There are 'n' equal intervals $[x_i - x_{i-1} = h]$

For the intervals the 'n' cubic spline equations are

$$y(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}\left[x_i - x\right]\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}\left[x - x_{i-1}\right]\left[y_i - \frac{h^2}{6}M_i\right], \text{ for } i = 1, 2, 3, \dots$$

Problems:

1. From the following table

x:	1	2	3
y:	-8	-1	18

Compute $y(1.5)$ and $y'(1)$ using Cubic Spline.

<u>Solution:</u>

Here $h=1$ [equally spaced]

$n=2$ [n = no. of intervals]

We know that,

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$$

Given, $M_0 = 0$, $M_n = 0$

$y_0 = -8$, $y_1 = -1$, $y_2 = 18$.

Here, $i = 1, 2, \dots, n-1$

Put $i=1$

Cubic spline for the interval $1 \le x \le 2$.

$$M_0 + 4M_1 + M_2 = \frac{6}{1^2}\left[y_0 - 2y_1 + y_2\right]$$

$$0 + 4M_1 + 0 = 6[-8 - 2(-1) + 18]$$

$$4M_1 = 6[-8 + 2 + 18]$$

$$4M_1 = 6[12]$$

$$M_1 = \frac{72}{4}$$

$$M_1 = 18$$

Cubic spline in $x_{i-1} < x < x_i$ is given by

$$y(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] +$$

$$\frac{1}{h}\left[(x_i - x)\right] + \frac{1}{h}\left[(x - x_{i-1})\right]\left[y_i - \frac{h^2}{6} M_i\right]$$

put $i=1$

$$= \frac{1}{6h}\left[(x_1 - x)^3 M_0 + (x - x_0)^3 M_1\right] + \frac{1}{1}\left[x_1 - x\right]\left[y_0 - \frac{1}{6}M_0\right]$$

$$+ \frac{1}{1}\left[x - x_0\right]\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left\{\left[x^3 - 3x^2 x_0 + 3x x_0^2 - x_0^3\right] M_1\right\} +$$

$$(x_1 - x)(y_0) + (x - x_0)\left[y_1 - \frac{1}{6}M_1\right]$$

$$= \frac{1}{6}\left[x^3 - 3\times1\times x^2 + 3\times1^2\times x - 1\right](18) + (2-x)(-8)$$

$$+ (x-1)\left[(-1) - \frac{1}{6}(18)\right]$$

$$= \frac{1}{6}\left[(x^3 - 3x^2 + 3x - 1)(18)\right] + \left[8x - 16\right] + \left[(x-1)(-4)\right]$$

$$= \frac{1}{6}\left[18x^3 - 54x^2 + 54x - 18 + 8x - 16\right] + \left[-4x + 4\right]$$

$$= \frac{1}{6}\left[18x^3 - 54x^2 + 62x - 34\right] - 4x + 4$$

$y(x) = 3x^3 - 9x^2 + 15x - 15.$

This is the cubic spline on the interval $1 \le x \le 2$.

$y(1.5) = 3(1.5)^3 - 9(1.5)^2 + 15(1.5) - 15.$

$\quad = 10.125 - 20.25 + 19.5 - 15$

$y(1.5) = -5.625$

$y'(x) = 9x^2 - 18x + 15$

$y'(1) = 9 - 18 + 15$

$y'(1) = 6.$

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2 Find the cubic spline approximation for the function $y = f(x)$ from the following data given $y_0'' = y_3'' = 0$ and find $y(0.5)$

x:	-1	0	1	2
y:	-1	1	3	35

<u>Solution:</u>

$h = 1$ [equally space]

$n = 3$ [n = no. of intervals]

Here $M_0 = 0$, $M_3 = 0$, $y_0 = -1$, $y_1 = 1$, $y_2 = 3$, $y_3 = 35$

We know that

$$M_{i-1} + 4M_i + M_{i+1} = \frac{6}{h^2}\left[y_{i-1} - 2y_i + y_{i+1}\right]$$

where $i = 1, 2, \ldots, n-1$

put $i=1$

$$M_0 + 4M_1 + M_2 = \frac{6}{1}[y_0 - 2y_1 + y_2]$$

$$0 + 4M_1 + M_2 = 6[-1 - 2(1) + 3]$$

$$4M_1 + M_2 = 0 \quad \text{—— (1)}$$

put $i=2$

$$M_1 + 4M_2 + M_3 = \frac{6}{1}[y_1 - 2y_2 + y_3]$$

$$M_1 + 4M_2 + M_3 = 6[1 - 2(3) + 35]$$

$$M_1 + 4M_2 + M_3 = 6[1 - 6 + 35]$$

$$M_1 + 4M_2 + 0 = 180 \quad \text{—— (2)}$$

Solve (1) and (2), we get

$$(1) \Rightarrow 4M_1 + M_2 = 0$$

$$-4M_1 + 16M_2 = 720$$

$$-15M_2 = -720$$

$$M_2 = 48$$

Substitute M_2 value in equation (1)

$$4M_1 + 48 = 0$$

$$4M_1 = -48$$

$$M_1 = -12$$

The cubic spline $x_{i-1} \le x \le x_{i+1}$ is given by

$$y(x) = \frac{1}{6h}\left[(x_i - x)^3 M_{i-1} + (x - x_{i-1})^3 M_i\right] + \frac{1}{h}(x_i - x)\left[y_{i-1} - \frac{h^2}{6}M_{i-1}\right] + \frac{1}{h}[x - x_{i-1}]\left[y_i - \frac{h^2}{6}M_i\right]$$

For $i=1$, $x_0 \le x \le x_1$

$$-1 \le x \le 0$$

Cubic spline in the interval $-1 \le x \le 0$

$M_0 = 0$, $M_1 = -12$, $M_2 = 48$, $M_3 = 0$

$x_0 = -1$, $x_1 = 0$, $x_2 = 1$, $x_3 = 2$

$y_0 = -1$, $y_1 = 1$, $y_2 = 3$, $y_3 = 35$

$= \frac{1}{6x_1}\left[(x_1-x)^3 M_0 + (x-x_0)^3 M_1\right] + \frac{1}{h}(x_1-x)\left(y_0 - \frac{1}{6}M_0\right)$

$+ \frac{1}{1}(x-x_0)\left[y_1 - \frac{1}{6}M_1\right]$

$= \frac{1}{6}\left[0 + (x+1)^3(-12)\right] + \frac{1}{1}\left[0 - x\right]\left[(-1) - \frac{1}{6}(0)\right] +$

$(x+1)\left[(1) - \frac{1}{6}(-12)\right]$

$= \frac{1}{6}\left[(x^3 + 3x^2 + 3x + 1)(-12)\right] + (x) + \left[(x+1)(1 + \frac{12}{6})\right]$

$= \frac{1}{6}\left[-12x^3 - 36x^2 - 36x - 12 + x + 3x + 3\right]$

$= \frac{1}{6}\left[-12x^3 - 36x^2 - 32x - 9\right]$

$y_1(x) = -2x^3 - 6x^2 - 2x + 1$

put $i = 2$

Cubic spline in the interval $0 \le x \le 1$

$y_2(x) = \frac{1}{6x_1}\left[(x_2 - x)^3 M_1 + (x - x_1)^3 M_2\right] + \frac{1}{1}(x_2 - x)\left(y_1 - \frac{1}{6}M_1\right)$

$+ \frac{1}{1}(x - x_1)\left(y_2 - \frac{1}{6}M_2\right)$

$= \frac{1}{6}\left[(1-x)^3(-12) + (x-0)^3(48)\right] + \frac{1}{1}(1-x)\left[1 - \frac{1}{6}(-12)\right]$

$+ (x-0)\left(3 - \frac{48}{6}\right)$

$= \frac{1}{6}\left[1 - 3x + 3x^2 - x^3(-12) + 48x^3\right] + (1-x)(1+2)$
$+ x(3-8)$

$= \left[-2 + 6x - 6x^2 + 2x^3 + 8x^3\right] + \left[-3x - 3x + 3\right] - 5x$

$= 2x^3 - 6x^2 + 6x + 8x^3 - 2 - 3x - 5x + 3$

$y_2(x) = 10x^3 - 6x^2 - 2x + 1$

put $i = 3$

$y_3(x) = \left[(x_3 - x)^3 M_2 + (x - x_2)^3 M_3\right] + \frac{1}{1}\left[(x_3 - x)(y_2 - \frac{1}{6}M_2)\right]$

$$+\frac{1}{1}\left[(x-x_2)\left(y_3-\frac{1}{6}M_3\right)\right]$$

$$=\frac{1}{6}\left[(2-x)^3(48)+(x-1)^3(0)\right]+(2-x)\left(3-\frac{1}{6}(48)\right)$$

$$+(x-1)\left(35-\frac{1}{6}(0)\right)$$

$$=\frac{1}{6}\left[(8-x^3-3(4)x+3(2)x^2)48\right]+(2-x)(-5)+35(x-1)$$

$$y_3(x)=64-8x^3-96x+48x^2-10-15x+35x-35$$

$$y_3(x)=-8x^3-96x+48x^2-15x+35x+64-10-35$$

$$y_3(x)=-8x+48x^2-56x+19$$

$$\therefore y(x)=\begin{cases}-2x^3-6x^2-2x+1, & -1\le x\le 0\\ 10x^3-6x^2-2x+1, & 0\le x\le 1\\ -8x^3+48x^2-56x+19, & 1\le x\le 2\end{cases}$$

$$y(0.5)\Rightarrow y_2(x)=10x^3-6x^2-2x+1,\quad 0\le x\le 1$$

$$=10(0.5)^3-6(0.5)^2-2(0.5)+1$$

$$=-0.25.$$