TOPIC : 2 – 2.10 – Newton's Forward & Backward interpolation

Newton's forward and backward difference formulas:

Newton's forward formula: [For equal intervals] (Δ):

Let $y=f(x)$ denote a function which takes the values $y_0, y_1, y_2, \dots, y_n$ corresponding to the values $x_0, x_1, \dots, x_n$ respectively of x. The value of 'x' are equi-distant.

Note:

The formula is used to interpolate the values of y nearer to the beginning value of the table.

Newton's backward Interpolation formula:

[For equal interval] (∇):

Let $y=f(x)$ denote a function which takes the values $y_0, y_1, \dots, y_n$ corresponding to the values $x_0, x_1, \dots, x_n$ respectively of x. The value of 'x' are equi-distant.

$$y(x) = y_n + \frac{n}{1!}\nabla y_n + \frac{n(n+1)}{2!}\nabla^2 y_n + \frac{n(n+1)(n+2)}{3!}\nabla^3 y_n + \cdots$$

where $n = \dfrac{x - x_n}{h}$

The formula is used to interpolate the values of y nearer to the end table.

Problems:

1\. Find the values of y at $x=21$ and $x=28$ from the following data

x:	20	23	26	29
$y(x)$:	0.3420	0.3907	0.4384	0.4848

Solution:

Difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$
20	0.3420			
		0.0487		
			−0.001	
				−0.0005
23	0.3907			
		0.0477		
			−0.0015	
26	0.4384			
		0.0464		
29	0.4848			

i) Newton's forward formula is.

$$y(x) = y_0 + \frac{n}{1!}\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \cdots$$

Here $n = \frac{x - x_0}{h}$, $x = 21$.

$$= \frac{21-20}{3} = \frac{1}{3} = 0.333$$

$$y(21) = 0.3420 + \frac{0.333}{1!}(0.0487) + \frac{(0.333)(0.3337)}{2!}(-0.001)$$

$$+ \frac{(0.333)(-0.6667)(-1.6667)}{3!}(-0.0005)$$

$y(21) = 0.3583$

ii) Newton's backward difference formula is.

$$n = \frac{x - x_n}{h} = \frac{28-29}{3} \quad (x = 28)$$

$$= -\frac{1}{3} = -0.3333$$

$$y(28) = 0.4848 + \frac{(-0.3333)}{1!}(0.0464) +$$

$$\frac{(-0.3333)}{2}(-0.0015) + \frac{(-0.3333)(0.667)(1.667)}{6}(-0.00$$

$$x_4 = \phi(x_3) = \frac{1}{\sqrt{x_3+1}} = \frac{1}{\sqrt{0.75767+1}} = \frac{1}{1.32577} = 0.75428$$

$$x_5 = \phi(x_4) = \frac{1}{\sqrt{x_4+1}} = \frac{1}{\sqrt{0.75428+1}} = \frac{1}{1.32449} = 0.75501$$

$$x_6 = \phi(x_5) = \frac{1}{\sqrt{x_5+1}} = \frac{1}{\sqrt{0.75501+1}} = \frac{1}{1.32477} = 0.75485$$

$$x_7 = \phi(x_6) = \frac{1}{\sqrt{x_6+1}} = \frac{1}{\sqrt{0.75485+1}} = \frac{1}{1.32477} = 0.75485$$

$\therefore$ The value of x_6 and x_7 are equal.

$\therefore$ The real root of the given equation is 0.75485.

2. Solve by iteration method $2x - \log_{10} x = 7$.

Solution:

Given $2x - \log_{10} x = 7$

$$2x - \log_{10} x - 7 = 0$$

Let $f(x) = 2x - \log_{10} x - 7$

$f(1) = 2(1) - \log_{10}(1) - 7 = -5 = -ve$

$f(2) = 2(2) - \log_{10}(2) - 7 = -3.3010 = -ve$

$f(3) = 2(3) - \log_{10}(3) - 7 = -1.4771 = -ve$

$f(4) = 2(4) - \log_{10}(4) - 7 = 0.39794 = +ve$

Hence a real root lies between 3 and 4.

Now, $2x - \log_{10} x - 7 = 0$ can be written as

$$2x = \log_{10} x + 7$$

$$x = \frac{1}{2}\left[\log_{10} x + 7\right]$$

$$\phi(x) = \frac{1}{2}\left[\log_{10} x + 7\right]$$

$$\phi(x) = \frac{1}{2}\left[\log_e x \cdot \log_{10} e + 7\right] \quad \left\{ \because \text{Change base rule } \log_{10} x = \log_e x \cdot \log_{10} e \right\}$$

$$\phi'(x) = \frac{1}{2}\left[\frac{1}{x} \cdot \log_{10} e + 7\right]$$

$$\phi'(x) = \frac{1}{2}\left[\frac{1}{x}(0.434)\right]$$

$$|\phi'(x)| = \left|\frac{1}{2}\left[\frac{1}{x}(0.434)\right]\right|$$

When $x = 3.7$ lies in $(3,4)$

$$|\phi'(3.7)| = \left|\frac{1}{2}\left[\frac{1}{3.7}(0.434)\right]\right|$$

$$= |0.0586| < 1.$$

$$\therefore |\phi'(3.7)| < 1$$

$|\phi'(x)| < 1$ lies in $(3,4)$

Let the initial approximation be $x_0 = 3.7$

$$x_1 = \phi(x_0) = \phi(3.7) = \frac{1}{2}\left[\log_{10}(3.7) + 7\right] = \frac{1}{2}[7.5682]$$

$$x_1 = 3.7841$$

$$x_2 = \phi(x_1) = \phi(3.7841) = \frac{1}{2}\left[\log_{10}(3.7841) + 7\right] = \frac{1}{2}[7.57796]$$

$$x_2 = 3.78898$$

$$x_3 = \phi(x_2) = \phi(3.78898) = \frac{1}{2}\left[\log_{10}(3.78898) + 7\right] =$$

$$\frac{1}{2}[7.57852] = 3.78926$$

$x_4 = \phi(x_3) = \phi(3.78926) = \frac{1}{2}\left[\log_{10}(3.78926) + 7\right]$

$= \frac{1}{2}\left[7.57855\right] = 3.78927$

$\therefore$ The value of x_3 and x_4 are equal.

$\therefore$ The real root is 3.78927.

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3. Find the real root of the equation $\cos x = 3x - 1$ using iteration method.

Solution:

Let $f(x) = \cos x - 3x + 1$

$f(0) = \cos(0) - 3(0) + 1 = 2 = +ve$

$f(1) = \cos(1) - 3(1) + 1 = -1.4597 = -ve$

Hence a real root lies between 0 and 1.

Now, $\cos x - 3x + 1 = 0$ can be written as

$\cos x + 1 = 3x$

$x = \frac{1}{3}\left[\cos x + 1\right]$

$\phi(x) = \frac{1}{3}\left[\cos x + 1\right]$

$\phi'(x) = \frac{1}{3}\left[-\sin x\right]$

$|\phi'(x)| = \left|\frac{1}{3}\sin x\right|$

When $x = 0.6$, $|\phi'(0.6)| = \left|\frac{1}{3}\sin(0.6)\right|$

$$= \left| \frac{1}{3}(0.0105) \right| = |0.1882| < 1$$

$\therefore |\phi'(x)| < 1$ in $(0,1)$

Let the initial approximation be $x_0 = 0.6$

$$x_1 = \phi(x_0) = \frac{1}{3}[\cos x_0 + 1] = \frac{1}{3}[\cos(0.6)+1]$$

$$x_1 = \frac{1}{3}[1.8253] = 0.6084$$

$$x_2 = \phi(x_1) = \frac{1}{3}[\cos x_1 + 1] = \frac{1}{3}[\cos(0.6084)+1]$$

$$x_2 = \frac{1}{3}[1.8206] = 0.6069$$

$$x_3 = \phi(x_2) = \frac{1}{3}[\cos(0.6069)+1] = \frac{1}{3}[1.8214] = 0.6071$$

$$x_4 = \phi(x_3) = \frac{1}{3}[\cos(0.6071)+1] = \frac{1}{3}[1.8213] = 0.6071$$

$\therefore$ The value of x_3 and x_4 are equal.

Hence the real root is 0.6071