TOPIC : 2.3- Joint distributions continuous Random variables

Continuous Random Variable

Joint Probability Density Function

If x and y are continuous random variables such that

$$P\left\{x - \frac{dx}{2} \le X \le x + \frac{dx}{2} \text{ and } y - \frac{dy}{2} \le Y \le y + \frac{dy}{2}\right\}$$

$$= f(x,y)\,dx\,dy$$

then we shall refer to $f(x,y)$ or f_{XY} as the joint probability function or joint probability density function of these two random variables provided

(i) $f(x,y) \ge 0$

(ii) $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

Joint Probability Distribution Function (or) Cumulative Distribution Function

The joint probability distribution function of a two dimensional random variables (X, Y) is denoted by $F_{XY}(x, y)$ and is given by

$$F_{XY}(x, y) = P[-\infty \le X \le x, -\infty \le Y = y]$$

$$= \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}(x, y)\, dx\, dy$$

Marginal Probability Density function

The marginal probability density function of the two random variables X and Y are defined as follows

$$f(x) = f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy$$

$$f(y) = f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx$$

Conditional Probability Density function

The conditional probability density function of x given y is given by

$$P[X = x / Y = y] = f(x/y) = \frac{f(x,y)}{f(y)}, \quad f(y) > 0$$

The conditional probability density function of y given x is given by

$$P[Y = y / X = x] = f(y/x) = \frac{f(x,y)}{f(x)}, \quad f(x) > 0$$

① The joint pdf of the random variable (x, y) is given by $f(x,y) = Kxy\,e^{-(x^2+y^2)}$, $x > 0$, $y > 0$. Find K.

WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$

$$\Rightarrow \int_0^{\infty}\int_0^{\infty} Kxy\,e^{-(x^2+y^2)}\,dx\,dy = 1$$

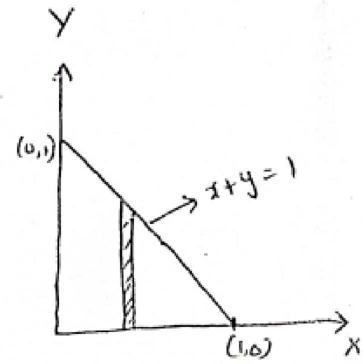
$$\Rightarrow K \int_0^{\infty} x e^{-x^2}\,dx \int_0^{\infty} y e^{-y^2}\,dy = 1$$

$$\Rightarrow K\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right) = 1 \Rightarrow \boxed{K = 4}$$

(2) If the joint pdf of (x, y) is $f(x,y) = \begin{cases} \frac{1}{4}, & 0<x,y<2 \\ 0, & \text{otherwise} \end{cases}$

Find $P(x+y \le 1)$.

Given $f(x,y) = \frac{1}{4}$, $0 < x, y < 2$



x varies from 0 to 1

y varies from 0 to $1-x$.

$$P[x+y \le 1] = \int_{x=0}^{1}\int_{y=0}^{1-x} \frac{1}{4}\, dy\, dx$$

$$= \frac{1}{4}\int_{x=0}^{1} (y)_0^{1-x}\, dx = \frac{1}{4}\int_{x=0}^{1} (1-x)\, dx$$

$$= \frac{1}{4}\left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{4}\left[1 - \frac{1}{2}\right] = \frac{1}{8}$$

(3)

Given the joint PDF of X and Y

$f(x,y) = C\,x(x-y)$, $0 \le x \le 2$, $-x \le y \le x$. (i) Find C (ii) find the marginal PDFs of X and Y (iii) find the conditional density of Y given X.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

$$\Rightarrow \int_{0}^{2}\int_{-x}^{x} c\,(x^2 - xy)\,dy\,dx = 1$$

$$\Rightarrow c\int_{0}^{2}\left[x^2y - x\frac{y^2}{2}\right]_{-x}^{x} dx = 1$$

$$\Rightarrow c\int_{0}^{2}\left[\left(x^3 - \cancel{\frac{x^3}{2}}\right) - \left(-x^3 - \cancel{\frac{x^3}{2}}\right)\right] dx = 1$$

$$\Rightarrow c\int_{0}^{2} 2x^3\,dx = 1 \Rightarrow c\,.\,2\left(\frac{x^4}{4}\right)_{0}^{2} = 1$$

$$\Rightarrow \frac{c}{2}(16) = 1 \Rightarrow \boxed{c = \frac{1}{8}}$$

$$f(x) = \int_{-\infty} f(x,y)\,dy = \int_{-x} \frac{1}{8}(x^2 - xy)\,dy$$

$$= \frac{1}{8}\left[x^2y - \frac{xy^2}{2}\right]_{-x}^{x} = \frac{1}{8}\left[\left(x^3 - \cancel{\frac{x^3}{2}}\right) - \left(-x^3 - \cancel{\frac{x^3}{2}}\right)\right]$$

$$= \frac{2x^3}{8} = \frac{x^3}{4}, \quad 0 < x < 2$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{2} \frac{1}{8}(x^2 - xy)\,dx$$

$$= \frac{1}{8}\left[\frac{x^3}{3} - \frac{x^2y}{2}\right]_{0}^{2} = \frac{1}{8}\left[\frac{8}{3} - 2y\right]$$

$$= \frac{1}{8}\left[\frac{8 - 6y}{3}\right] = \frac{1}{12}(4 - 3y), \quad -2 < y < 2$$

$$f(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{0}^{2} \frac{1}{8}(x^2 - xy)\,dx$$

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conditional density function of Y gn. X

$$f(y/x) = \frac{f(x,y)}{f(x)} = \frac{\frac{1}{8}x(x-y)}{\frac{x^3}{4}}$$

$$= \frac{x-y}{2x^2}, \quad \begin{matrix} 0 < x < 2 \\ -x < y < -x \end{matrix}$$

(4) Find the value of K, if $f(x,y) = K(1-x)(1-y)$, $0 < x, y < 1$ is to be the joint density function.

WKT $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$

$\Rightarrow \int_0^1\int_0^1 K(1-x)(1-y)\, dx\, dy = 1$

$\Rightarrow K \int_0^1 (1-x)\, dx \int_0^1 (1-y)\, dy = 1$

$\Rightarrow K \left[x - \frac{x^2}{2}\right]_0^1 \left[y - \frac{y^2}{2}\right]_0^1 = 1$

$\Rightarrow K \left[1 - \frac{1}{2}\right]\left[1 - \frac{1}{2}\right] = 1$

$\Rightarrow K\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1 \Rightarrow \boxed{K = 4}$