Topic : Runge Kutta Method

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$$y_1 = 1+(0.1)\, f\left[0+\frac{0.1}{2},\ 1+\frac{0.1}{2}(1)\right]$$
$$= 1+(0.1)\, f(0.05, 1.05)$$
$$= 1+(0.1)\,(1.105)$$
$$= 1.1105$$
$$y(0.1) = 1.1105.$$

Fourth order Runge Kutta Method for Solving first and Second order Equations

Second order RK method.

If the initial values of (x, y) for the differential equation $\frac{dy}{dx} = f(x,y)$ then the first increment in y namely Δy is calculated from the formula.

$$K_1 = h\, f(x, y)$$
$$K_2 = h\, f\left(x+\frac{h}{2},\ y+\frac{K_1}{2}\right)$$
$$\Delta y = K_2 \quad \text{where } h = \Delta x.$$

Third order RK Method

$$K_1 = h f(x, y)$$
$$K_2 = h\, f\left(x+\frac{h}{2},\ y+\frac{K_1}{2}\right)$$
$$K_3 = h\, f(x+h,\ y+2K_2-K_1)$$

and $\Delta y = \frac{1}{6}(K_1 + 4K_2 + K_3)$

Fourth order RK method:

$$K_1 = h f(x,y)$$
$$K_2 = h f(x+\frac{h}{2}, y+\frac{K_1}{2})$$
$$K_3 = h f(x+\frac{h}{2}, y+\frac{K_2}{2})$$
$$K_3 = h f(x+h, y+K_3)$$

and $\Delta y = \frac{1}{6}[K_1+2K_2+2K_3+K_4]$.

$$y(x+h) = y(x) + \Delta y.$$

Example: Given $\frac{dy}{dx} = x^2+y$, $y(0)=2$ Compute $y(0.2)$, $y(0.4)$ and $y(0.6)$ by Runge Kutta Method of Fourth order.

Solution: given $y' = x^2+y$, $y(0)=2$

(ie) $x_0=0$, $y_0=2$, $x_1=0.2$, $x_2=0.4$, $x_3=0.6$

Fourth order RK algorithm.

$$K_1 = h f(x,y)$$
$$K_2 = h f(x+\frac{h}{2}, y+\frac{K_1}{2})$$
$$K_3 = h f(x+\frac{h}{2}, y+\frac{K_2}{2})$$
$$K_4 = h f(x+h, y+K_3)$$
$$\Delta y = \frac{1}{6}[K_1+2K_2+2K_3+K_4]$$
$$y(x+h) = y(x) + \Delta y.$$

⑦

$K_1 = h\, f(x_0, y_0)$

$= (0.2)\,[x_0^3 + y_0]$

$= (0.2)\,[0+2] = 0.4$

$K_2 = h\, f\left[x_0 + \frac{h}{2},\, y_0 + \frac{K_1}{2}\right]$

$= (0.2)\, f\left[0 + \frac{0.2}{2},\, 2 + \frac{0.4}{2}\right]$

$= (0.2)\, f\,[0.1,\, 2.2]$

$= (0.2)\,[(0.1)^3 + 2.2]$

$= 0.4402$

$K_3 = h\, f\left[x_0 + \frac{h}{2},\, y_0 + \frac{K_2}{2}\right]$

$= (0.2)\, f\left[0 + \frac{0.2}{2},\, 2 + \frac{0.4402}{2}\right]$

$= (0.2)\, f\,[0.1,\, 2.2201]$

$= (0.2)\,[(0.1)^3 + 2.2201]$

$= 0.44422$

$K_4 = h\, f(x_0 + h,\, y_0 + K_3)$

$= (0.2)\, f\,[0 + 0.2,\, 2 + 0.44422]$

$= (0.2)\, f\,[0.2,\, 2.44422]$

$= (0.2)\,[(0.2)^3 + 2.44422]$

$K_4 = 0.490444.$

$\Delta y = \frac{1}{6}\,[K_1 + 2K_2 + 2K_3 + K_4]$

$= \frac{1}{6}\,[0.4 + 2(0.4402) + 2(0.44422) + 0.490444]$

$= 0.44821.$

$y(0.2) = y_1 = y_0 + \Delta y = 2 + 0.44321 = 2.44321$

$\therefore \boxed{y(0.2) = 2.4432}$

<u>Second iteration</u>.

$K_1 = h\, f(x_1, y_1)$

$= (0.2)\, f[0.2, 2.443]$

$= (0.2)[(0.2)^3 + 2.443]$

$= 0.4902$

$K_2 = h\, f(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2})$

$= (0.2)\, f[0.2 + \frac{0.2}{2}, 2.443 + \frac{0.4902}{2}]$

$= (0.2)[(0.3)^3 + 2.6881]$

$= 0.5430.$

$K_3 = h\, f(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2})$

$= (0.2)\, f[0.2 + \frac{0.2}{2}, 2.443 + \frac{0.5430}{2}]$

$= (0.2)[(0.3)^3 + 2.7145]$

$= 0.5483.$

$K_4 = h\, f[x_1 + h, y_1 + K_3]$

$= (0.2)\, f[0.2 + 0.2, 2.443 + 0.5483]$

$= (0.2)[(0.4)^3 + 2.9913]$

$= 0.6111.$

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$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= \frac{1}{6}[0.4902 + 2(0.543) + 2(0.5483) + 0.6111]$$

$$\Delta y = 0.5473.$$

$$y(0.4) = y_2 = y_1 + \Delta y = 2.443 + 0.5473 = 2.99$$

$$\boxed{y(0.4) = 2.99}$$

<u>Third iteration</u>:

Here $x_2 = 0.4,\ y_2 = 2.99.$

$$k_1 = hf(x_2, y_2)$$

$$= (0.2)\, f[0.4, 2.99]$$

$$= (0.2)[(0.4)^3 + 2.99]$$

$$= (0.2)(3.054) = 0.6108$$

$$k_2 = hf(x_2 + \tfrac{h}{2},\ y_2 + \tfrac{k_1}{2})$$

$$= (0.2)\, f[0.4 + \tfrac{0.2}{2},\ 2.99 + \tfrac{0.6108}{2}]$$

$$= (0.2)[(0.5)^3 + 3.2954]$$

$$k_2 = 0.6841$$

$$k_3 = hf(x_2 + \tfrac{h}{2},\ y_2 + \tfrac{k_2}{2})$$

$$= (0.2)\, f[0.4 + \tfrac{0.2}{2},\ 2.99 + \tfrac{0.6841}{2}]$$

$$= (0.2)[(0.5)^3 + 3.3321]$$

$$= 0.6914.$$

$$K_4 = hf(x_2+h,\ y_2+K_3)$$

$$= (0.2) f[0.4+0.2,\ 2.99+0.6914]$$

$$= (0.2)\left[(0.6)^3 + 3.68.14\right]$$

$$= 0.7795$$

$$\Delta y = \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= \frac{1}{6}\left[0.6108 + 2(0.6841) + 2(0.6914) + 0.7795\right]$$

$$= \frac{1}{6}\left[4.1413\right]$$

$$= 0.6902$$

$$y(0.6) = y_3 = y_2 + \Delta y = 2.99 + 0.6902 = 3.68.$$

$$\therefore \boxed{y(0.6) = 3.68}$$

<u>Example</u>: using RK method of fourth order.

Solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2$.

<u>Solution</u>. given $y' = \frac{y^2-x^2}{y^2+x^2}$, $x_0=0,\ x_1=0.2,\ y_0=1$

$$K_1 = hf(x_0, y_0)$$

$$= (0.2) f(0,1)$$

$$= (0.2)\left(\frac{1-0}{1+0}\right) = 0.2$$

$$K_2 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_1}{2}\right)$$

$$= (0.2)\, f\left[0 + \frac{0.2}{2},\ 1 + \frac{0.2}{2}\right]$$

$$= (0.2)\, f\left[0.1,\ 1.1\right]$$

$$= (0.2)\left[\frac{(1.1)^2 - (0.1)^2}{(1.1)^2 + (0.1)^2}\right]$$

$$= (0.2)\,(0.9836) = 0.19672$$

$$K_3 = hf\left(x_0 + \frac{h}{2},\ y_0 + \frac{K_2}{2}\right)$$

$$= (0.2)\, f\left[0 + \frac{0.2}{2},\ 1 + \frac{0.19672}{2}\right]$$

$$= (0.2)\, f\left[0.1,\ 1.19672\right)$$

$$= (0.2)\left[\frac{(1.9672)^2 - (0.1)^2}{(1.9672)^2 + (0.1)^2}\right]$$

$$= 0.1967$$

$$K_4 = hf\left(x_0 + h,\ y_0 + K_3\right)$$

$$= (0.2)\, f\left(0 + 0.2,\ 1 + 0.1967\right)$$

$$= (0.2)\left[0.2,\ 1.1967\right)$$

$$= (0.2)\left[\frac{(1.1967)^2 - (0.2)^2}{(1.1967)^2 + (0.2)^2}\right]$$

$$= 0.1891$$