TOPIC : 4 NEWTON'S RAPHSON method

Newton's method (or) Newton - Raphson method:

Formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Condition for convergence of Newton - Raphson method is

$$|f(x) \cdot f''(x)| < |f'(x)|^2.$$

Rate of Convergence of Newton's method is 2

(or)

The order of convergence of Newton's method is 2

Problems:

1. Find the positive root of $x^4 - x = 10$ correct to three decimal places using Newton's Raphson method.

Solution:

Let $f(x) = x^4 - x - 10$

$f'(x) = 4x^3 - 1$

$f(1) = (1)^4 - 1 - 10 = -10 = -ve$

$f(2) = (2)^4 - 2 - 10 = 16 - 2 - 10 = 4 = +ve$

Hence the root lies between 1 and 2.

$\therefore |f(1)| = 10$

$|f(2)| = 4$

$\therefore |f(1)| > |f(2)|$

$\therefore$ The root is nearer to 2.

Let we take $x_0 = 2$.

Newton's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

put $n = 0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

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$f(x_0) = f(2) = (2)^4 - 2 - 10 = 16 - 2 - 10 = 4$

$f'(x_0) = f'(2) = 4(2)^3 - 1 = 32 - 1 = 31$

$x_1 = 2 - \frac{4}{31}$

$x_1 = 2 - 0.1290 = 1.871.$

put $n = 1$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

$= 1.871 - \frac{f(1.871)}{f'(1.871)}$

$f(1.871) = (1.871)^4 - 1.871 - 10 = 12.254 - 1.871 - 10$
$= 0.883.$

$f'(1.871) = 4(1.871)^3 - 1 = 26.199 - 1 = 25.199$

$x_2 = 1.871 - \frac{0.383}{25.199} = 1.871 - 0.015 = 1.856$

$x_2 = 1.856.$

put $n = 2$

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$

$= 1.856 - \frac{f(1.856)}{f'(1.856)}$

$f(1.856) = (1.856)^4 - 1.856 - 10 = 0.010$

$f'(1.856) = 4(1.856)^3 - 1 = 25.574$

$$x_3 = 1.856 - \frac{0.010}{25.574}$$

$$= 1.856.$$

∴ The value of x_2 and x_3 are equal.

Hence the real root is 1.856.

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2. Find a root of $x\log_{10}^{x} - 1.2 = 0$ by Newton - Raphson method correct to 3-decimal places.

Solution:

Let $f(x) = x\log_{10}^{x} - 1.2$

$f(x) = x[\log_e^x \cdot \log_{10}^e] - 1.2$ [using change base rule $\log_{10}^x = \log_e^x \cdot \log_{10}^e$]

$f(x) = \log_{10}^e[x\log_e^x] - 1.2$

$f'(x) = \log_{10}^e[x\cdot\frac{1}{x} + \log_e^x]$

$f'(x) = \log_{10}^e[1 + \log_e^x]$

$f'(x) = \log_{10}^e + \log_{10}^e \log_e^x = \log_{10}^e + \log_{10}^x$

$f(1) = 1 \cdot \log_{10}^1 - 1.2 = -1.2 = -ve$

$f(2) = 2 \cdot \log_{10}^2 - 1.2 = -0.598 = -ve$

$f(3) = 3\log_{10}3 - 1.2 = 0.231 = +ve$

Hence the root lies between 2 and 3

∴ $|f(2)| = 0.598$.

∴ $|f(3)| = 0.231$

$|f(2)| > |f(3)|$

∴ The root is nearer to 3

Let $x_0 = 3$

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Newton's formula is,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

put $n=0$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 3 - \frac{f(3)}{f'(3)}$$

$f(3) = 3\log_{10} 3 - 1.2 = 0.231$, $f'(3) = \log_{10} e + \log_{10} 3 = 0.9111$

$$x_1 = 3 - \left(\frac{0.231}{0.9111}\right)$$

$$x_1 = 3 - 0.2535 = 2.747$$

put $n=1$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 2.747 - \frac{f(2.747)}{f'(2.747)}$$

$f(x_1) = f(2.747) = 2.747 \log_{10}(2.747) - 1.2 = 0.0055$

$f'(x_1) = \log_{10} e + \log_{10}(2.747) = 0.8729$

$$x_2 = 2.747 - \left(\frac{0.0055}{0.8729}\right) = 0.2741$$

put $n=2$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= (2.741) - \frac{f(2.741)}{f'(2.741)}$$

$f(2.741) = (2.741)\log_{10}(2.741) - 1.2 = 0.0003$

$f'(2.741) = \log_{10} e + \log_{10}(2.741) = 0.434 + 0.4379 = 0.8719$

$$x_3 = 2.741 - \left(\frac{0.0003}{0.8719}\right)$$

$$x_3 = 2.741$$

$\therefore$ The value of x_2 and x_3 are equal.

$\therefore$ The real root is 2.741.

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3. Find the iterative formula for finding the value of $\frac{1}{N}$ where N is a real x_0 using Newton's Raphson method. Hence evaluate $1/26$ correct to 4 decimal place.

Solution:

$$\text{Let } x = 1/N$$

$$N = 1/x$$

$$1/x - N = 0$$

$$\text{Let } f(x) = 1/x - N$$

$$f(x) = x^{-1} - N$$

$$f'(x) = (-1)\, x^{-1-1} - 0 = -x^{-2} = -1/x^2$$

Newton's formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{(1/x_n - N)}{(-1/x_n^2)}$$

$$= x_n + x_n^2 (1/x_n - N)$$

$$= x_n + x_n - x_n^2 N$$

$$x_{n+1} = 2x_n - x_n^2 N = x_n(2 - x_n N)$$

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$\therefore x_{n+1} = x_n(2 - x_n N)$ is a iterative formula.

To find $1/26$:

Now, $1/N = 1/26$ Take $N = 26$

Let $x_0 = 1/26$ put $n = 0$

$x_1 = x_0(2 - x_0 N) = \frac{1}{26}\left(2 - \frac{1}{26} \cdot 26\right) = \frac{1}{26}(2-1)$

$= 1/26 (1) = 0.0385.$

put $n = 1$

$x_2 = x_1(2 - x_1 N)$

$= (0.0385)[2 - 0.0385(26)]$

$x_2 = 0.0385.$

$\therefore$ The value of x_1 and x_2 are equal.

Hence the value of $1/26 = 0.0385.$

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4. Derive the newton's algorithm for finding the p^{th} root of a number N and hence find the cube root of 17.

Solution: Given $x = \sqrt[p]{N}$

$x = (N)^{1/p}$

$x^p = N$

$x^p - N = 0$

$f(x) = x^p - N$

$f'(x) = px^{p-1}$

By Newton's formula,

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$= x_n - \frac{(x_n^p - N)}{px_n^{p-1}}$

$$= \frac{x_n\, p\, x_n^{p-1} - x_n^{p} + N}{p x_n^{p-1}} = \frac{x_n^{1+p-1} \cdot p - x_n^{p} + N}{p x_n^{p-1}}$$

$$x_{n+1} = \frac{x_n^{p} \cdot p - x_n^{p} + N}{p x_n^{p-1}} = \frac{x_n^{p}(p-1) + N}{p x_n^{p-1}}$$

To find $\sqrt[3]{17}$:

The value of $\sqrt[3]{17}$ is 2.5713

Hence the root lies between 2 and 3.

Let take $x_0 = 2$ and $p = 3$.

$\sqrt[3]{17} = \sqrt[p]{N}$

put $n=0$

$$x_1 = \frac{(p-1)\, x_0^{p} + N}{p x_0^{p-1}} = \frac{(3-1)(2)^3 + (17)}{(3)(2)^{3-1}}$$

$$x_1 = \frac{2(8)+17}{3(2)^2} = 2.75$$

put $n=1$

$$x_2 = \frac{(p-1)\, x_1^{p} + N}{p x_1^{p-1}} = \frac{(3-1)\,(2.75)^3 + 17}{(2.75)^{3-1} \cdot 3}$$

$$x_2 = \frac{(2)(20.7969)+17}{3(2.75)^2} = 2.5826$$

put $n=2$

$$x_3 = \frac{(p-1)\,(2.5826)^3 + 17}{3(2.5826)^{3-1}} = \frac{2(17.2255)+17}{20.0095}$$

$$x_3 = \frac{51.451}{20.0095} = 2.5713$$

put $n=3$

$$x_4 = \frac{(3-1)\,(2.5713)^3 + 17}{3(2.5713)^{3-1}} = \frac{2(2.5713)^3 + 17}{3(2.5713)^2}$$